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ASYMPTOTIC ANALYSIS OF LONG-TERM INVESTMENT WITH TWO ILLIQUID AND CORRELATED ASSETS

XINFU CHEN   MIN DAI   WEI JIANG   CONG QIN

Abstract. We consider a long-term portfolio choice problem with two illiquid and correlated assets and formulate it as an eigenvalue problem in the form of a variational inequality. The eigenvalue is associated with the portfolio's optimal long-term growth rate, and the free boundaries implied by the variational inequality correspond to the optimal trading strategy. After proving the existence and uniqueness of viscosity solutions for the eigenvalue problem, we perform an asymptotic expansion in terms of small correlations and obtain semi-analytical approximations of the free boundaries and the optimal growth rate. Our leading order expansion implies that the free boundaries are orthogonal to each other at four corners and have C^1 regularity. We propose an efficient numerical algorithm based on the expansion, which proves to be accurate even for large correlations and transaction costs. Moreover, following the approximate trading strategy, the resulting growth rate is very close to the optimal one.

Keywords: Transaction costs, Asymptotic expansion, Multiple assets, Correlation

AMS Classification: 91G10, 93E20, 41A60

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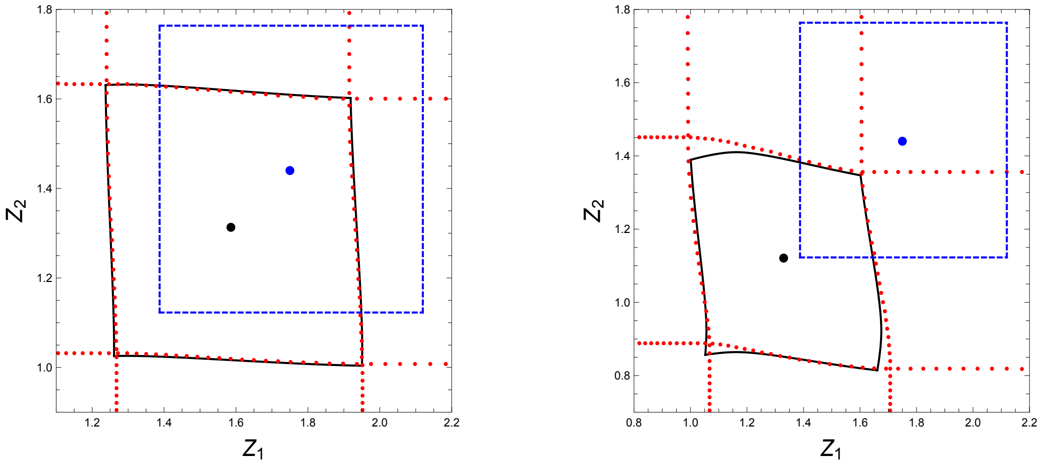

Figure 1. The No-transaction Regions with Different Correlations (Left: $\rho = 0.1$, Right: $\rho = 0.3$). In each plot, the thick black (resp. blue) dot is the Merton's strategy in the absence of transaction costs but with a non-zero (resp. zero) correlation; The blue dashed rectangle is the uncorrelated case (i.e., $\rho = 0$) but with transaction costs; The red dotted lines are numerical solution in the presence of both correlation and transaction costs; The black solid lines are the asymptotic expansion solution in the presence of both correlation and transaction costs. Other parameters are summarized in Table 2.

1. Introduction

Merton (1969, 1971) pioneers in continuous time portfolio selection problems. Magill and Constantinides (1976) introduce transaction costs to Merton's model and show that a no-transaction region exists. In this paper, we consider a long-term investment problem for a constant absolute risk aversion (CARA) investor who faces proportional transaction costs and has access to two *correlated* risky assets as well as a riskfree asset. If the risky assets are uncorrelated, Liu (2004) shows that the problem can be reduced, by virtue of the separability of the CARA utility function, to the single risky asset case, which leads to the separability of the optimal investment strategy, i.e., keeping the dollar amount invested in each risky asset between two constant levels. Graphically, the no-transaction region is a rectangle in the two risky assets plane for the uncorrelated case; see the blue dashed lines in Figure 1. However, the separability loses effect when risky assets are correlated. We aim to employ asymptotic analysis to investigate how the correlation of two risky assets affects the optimal trading strategy and the portfolio's long-term growth rate.

In the presence of proportional transaction costs, the long-term investment problem can be formulated as an eigenvalue problem in the form of a variational inequality with gradient constraints, where the eigenvalue represents the long-term excess growth rate (i.e., the long-term growth rate minus the riskfree rate), and the free boundaries implied by the variational inequality correspond to the optimal trading strategy. Due to the lack of analytical solutions, we provide a theoretical proof for the existence and uniqueness of viscosity solutions to the eigenvalue problem. To characterize the optimal trading strategy, one needs to locate the free boundaries. Despite numerical methods (e.g., the finite difference method) can be used, a large amount of computation is required to accurately identify the free boundaries as they are associated with the gradient of the corresponding value function. Moreover, numerical methods are likely unstable when transaction costs are tiny. In contrast, our asymptotic expansion gives a semi-analytical approximation solution, which efficiently approximates the optimal trading

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strategy. The asymptotic expansion also allows us to quickly obtain a reference of the long-term excess growth rate. Moreover, numerical experiments reveal that following the approximate trading strategy, the resulting growth rate is very close to the optimal one. Besides, asymptotic analysis allows us to study the behavior of optimal trading strategy qualitatively. In particular, the leading order in our asymptotic expansion shows that, compared to the uncorrelated case, the no-transaction region is no longer a rectangle but a quadrangle with curved boundaries being *orthogonal* to each other at four corners; see Figure 1. This orthogonality implies that the optimal trading boundaries have certain regularity, which sheds light on the future study on the regularity of the trading boundaries in high dimensional cases.

Related Literature. This paper adds to a large body of literature on portfolio selection with transaction costs. Since the seminal work in Magill and Constantinides (1976), Merton's model with transaction costs has been extensively studied along different lines, e.g., theoretical characterization of the no-transaction region (Davis and Norman (1990), Shreve and Soner (1994), Liu and Loewenstein (2002), Dai and Yi (2009), Dai et al. (2009), Chen and Dai (2013)), the effect of transaction costs on liquidity premium (Constantinides (1986), Jang et al. (2007)), utility indifference pricing (Davis et al. (1993), Constantinides and Zariphopoulou (2001)), martingale approach (Cvitanic and Karatzas (1996)), shadow prices (Kallsen and Muhle-Karbe (2010)), numerical solutions (Gennotte and Jung (1994), Muthuraman (2006), Muthuraman and Kumar (2006), Dai and Zhong (2010)), risk-sensitive asset management (Bielecki and Pliska (2000), Bielecki et al. (2004)), and asymptotic analysis.

This paper is closely related to the extensive literature on asymptotic analysis with small transaction costs that has gained a lot of attention since the early important contributions by Shreve and Soner (1994), Atkinson and Wilmott (1995), Whalley and Wilmott (1997), and Janeček and Shreve (2004). Recently, in a general market setting, Kallsen and Muhle-Karbe (2017) formally derive leading-order optimal trading policies. Melnyk et al. (2019) show that the leading order for small transaction costs is the same for agents with additive utilities and agents with recursive utilities. For a general utility, Soner and Touzi (2013), Possamaï et al. (2015), and Altarovivi et al. (2015) apply homogenization and the viscosity solution technique to get rigorous expansions in multi-asset case with proportional transaction costs and fixed transaction costs, respectively. Melnyk and Seifried (2018) consider a long-term investment problem in an incomplete market for both proportional transaction costs and Morton-Pliska costs. In contrast to this strand of literature, we conduct asymptotic analysis with small correlations instead of small transaction costs.

There is very little literature on asymptotic analysis with small correlations. To the best of our knowledge, Atkinson and Ingpochai (2006) may be the only one in which a perturbation method is used to examine a multiple-asset portfolio optimization problem with small correlations and small proportional transaction costs. The differences between theirs and our paper lie in two aspects: (a) They consider a life-time investment and consumption problem for a constant relative risk aversion (CRRA) investor. In contrast, we consider a long-term investment for a CARA investor and need to solve an eigenvalue problem as a result. (b) Their expansion relies on small correlations and small transaction costs, while our expansion relies on small correlations only. Thus, our expansion works even with large transaction costs; see Figure 4.

The rest of the paper is organized as follows. In the next section, we present the problem formulation and our main theoretical results, including the existence and uniqueness of viscosity solutions to the eigenvalue problem and an asymptotic expansion with small correlations. In

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Section 3, we conduct an extensive numerical analysis to demonstrate our expansion and investigate the impact of correlation and transaction costs on the optimal trading strategy. The proofs of Theorem 1 and Theorem 2 are presented in Section 4 in Section 5, respectively. A heuristic derivation of the eigenvalue problem, together with some basic properties in the one risky-asset case, is relegated to Appendix.

2. Problem Formulation and Main Results

2.1. Problem Formulation

We consider a financial market consisting of a risk-free asset with a constant interest rate $r \geq 0$ and two *correlated* risky assets. The price dynamics of the i-th risky asset $(i = 1, 2)$ is assumed to follow a geometric Brownian motion, i.e.,

$$dP_t^i/P_t^i = \alpha_i dt + \sigma_i dW_t^i,$$

where $\alpha_i \in \mathbb{R}$ and $\sigma_i > 0$ are the constant expected return rate and volatility of the i-th risky asset, respectively, and W^1 and W^2 are two standard Brownian motions with a constant correlation coefficient $\rho \in (-1, 1)$. We further assume that trading in the risk-free asset incurs no transaction costs, while there are proportional transaction costs for trading the risky assets. To be more precise, let L_t^i and M_t^i be respectively the cumulative purchase and sales (in dollars) of the i-th risky asset. Then, during a time period $[t, t + dt)$, a transfer of money from the bank account to the i-th risky asset incurs purchasing costs $\lambda_i dL_t^i$. Similarly, there are selling costs $\mu_i dM_t^i$ during the time period $[t, t + dt)$. Here $\lambda_i \geq 0$ and $\mu_i \in [0, 1)$ are the constant proportions of transaction costs for purchasing and selling the i-th risky asset, respectively. Therefore, given an initial allocation $(x, y) = (x, y_1, y_2) \in \mathbb{R}^3$ in risk-free and risky assets and a trading strategy $(L, M) = (\{L^1, L^2\}, \{M^1, M^2\})$, the dollar amounts in the risk-free asset and the two risky assets, denoted by X_t, Y_t^1, and Y_t^2, respectively, evolve according to

$$\begin{cases} dX_t = rX_t dt - \displaystyle\sum_{i=1}^{2} (1 + \lambda_i) dL_t^i + \sum_{i=1}^{2} (1 - \mu_i) dM_t^i, \quad X_{0^-} = x, \\ dY_t^i = Y_t^i \Big(\alpha_i dt + \sigma_i dW_t^i \Big) + dL_t^i - dM_t^i, \quad Y_{0^-}^i = y_i \quad \text{for } i = 1, 2. \end{cases} \tag{2.1}$$

Following Guasoni and Muhle-Karbe (2015), we consider a CARA utility investor (i.e., $-e^{-\nu z}$ for $z \in \mathbb{R}$ with the absolute risk aversion $\nu > 0$), who maximizes the long-term certainty equivalent annuity[1]

$$\sup_{(L,M) \in \mathcal{C}(x,y)} \liminf_{T \to \infty} -\frac{1}{T} \ln \mathbb{E}^{x,y} \Big[\exp \Big(-\nu \Big\{ X_T + \ell(Y_T) - (x + y \cdot \mathbf{1}) e^{rT} \Big\} \Big) \Big] \tag{2.2}$$

subject to (2.1), where $\mathbf{1} = (1, 1)$, $\ell(y)$ is the liquidated wealth, namely,

$$\ell(y) = \sum_{i=1}^{2} \ell_i(y_i), \quad \ell_i(y_i) = \begin{cases} (1 - \mu_i) y_i & \text{if } y_i \geq 0, \\ (1 + \lambda_i) y_i & \text{if } y_i < 0, \end{cases} \tag{2.3}$$

and $\mathcal{C}(x, y)$ is the set of all admissible controls such that the above system of stochastic differential equations (2.1) has a unique strong solution with the initial state $(X_{0^-}, Y_{0^-}) = (x, y)$

[1] The formulation is similar to that given in Definition 2.2 of Guasoni and Muhle-Karbe (2015). As noted in Guasoni and Muhle-Karbe (2015), when $r > 0$, the risky investment part grows linearly with the horizon T, while the risk-free part grows exponentially at the risk-free rate. Hence, the long-term certainty equivalent annuity defined in (2.2) can be interpreted as the linear growth part contributed by the risky investment; see the subsequent discussion for the special case without transaction costs, i.e., the classic Merton (1969) problem.

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and that $\mathbb{E}\int_0^T |Y_t e^{-r\nu(X_t+Y_t)}|^2 dt < \infty$ for all $T > 0$ to rule out doubling strategies. This is an ergodic control problem.

Denote by $z = \nu e^{rT} y$ the investment in risky assets adjusted by interest rate and risk aversion. The ergodic control problem (2.2) turns out to be associated with the following eigenvalue problem with the variational inequality form[2]

$$\min\{\theta - \mathcal{A}[u],\ \mathcal{B}[u],\ \mathcal{S}[u]\} = 0 \qquad \text{in } \mathbb{R}^2, \tag{2.4}$$

where the differential operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$ are defined, respectively, as

$$\mathcal{A}[u] = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_{ij}z_i z_j[u_{z_i z_j} - u_{z_i}u_{z_j}] + \sum_{i=1}^{2}(\alpha_i - r)z_i u_{z_i}, \tag{2.5}$$

$$\mathcal{B}[u] = \min\{1+\lambda_1 - u_{z_1},\ 1+\lambda_2 - u_{z_2}\}, \tag{2.6}$$

$$\mathcal{S}[u] = \min\{-1+\mu_1 + u_{z_1},\ -1+\mu_2 + u_{z_2}\}, \tag{2.7}$$

and

$$\Sigma = (\sigma_{ij})_{2\times 2} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{2.8}$$

In (2.4), the eigenvalue θ relates to the long-term certainty equivalent annuity, and the eigenfunction u characterizes the optimal policy.

Before proceeding, let us recall the classic results in Merton (1969) in the absence of transaction costs, which is very helpful to understand our further analysis. It is well-known that for a finite horizon investment with exponential utility, Merton's solution has an explicit form

$$\overline{V}(x,y,t;T) = -\exp\Big(-\nu(x+y\cdot\mathbf{1})e^{r(T-t)} - \frac{1}{2}(\alpha-\mathbf{r})^{tr}\Sigma^{-1}(\alpha-\mathbf{r})(T-t)\Big),$$

and the corresponding optimal allocation in risky assets is given by

$$\frac{\Sigma^{-1}(\alpha-\mathbf{r})}{\nu e^{r(T-t)}},$$

where $\alpha = (\alpha_1,\alpha_2)^{tr}$, $\mathbf{r} = (r,r)^{tr}$, Σ is as given by (2.8), and M^{tr} stands for the transpose of a matrix M. By removing the exponential growth term $\nu(x+y\cdot\mathbf{1})e^{r(T-t)}$, the long-term certainty equivalent annuity contributed by risky investment is given by

$$\overline{\theta} = \frac{1}{2}(\alpha-\mathbf{r})^{tr}\Sigma^{-1}(\alpha-\mathbf{r}). \tag{2.9}$$

We call

$$\overline{\pi} = \Sigma^{-1}(\alpha-\mathbf{r}) \tag{2.10}$$

the Merton's strategy (deflated by risk aversion and interest rate) for the long-term problem (2.2). Interestingly, one can easily check[3] that $\overline{\theta}$ is nothing but the portfolio's long-term excess growth rate (i.e., the long-term growth rate nets of interest rate)[4], i.e.,

$$\overline{\theta} = \sup_{\pi}\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}[\ln(X_T+Y_T)] - r.$$

Hence, we may also interpret θ as the long-term excess growth rate in the presence of proportional transaction costs.

[2] See Appendix A for a heuristic derivation.

[3] See, e.g., Melnyk and Seifried (2018).

[4] It is worth pointing out that this coincidence between $\overline{\theta}$ and the long-term excess growth rate holds for the exponential utility and usually fails for a general utility function. In addition, the formulation of long-term growth rate implies a no-bankruptcy assumption that $X_T + Y_T > 0$, which is not required in (2.2).

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2.2. **Main Results**

First, we prove the existence and uniqueness of viscosity solutions to the eigenvalue problem (2.4) and characterize the resulting optimal trading strategy.

Theorem 1. *Assume that r, ρ, λ_i, μ_i, α_i, and σ_i satisfy the following conditions:*

$$r \geq 0, \quad |\rho| \leq 1, \quad \lambda_i \geq 0, \quad \mu_i \in [0,1), \quad \lambda_i + \mu_i > 0, \quad \alpha_i \neq r, \quad \sigma_i > 0 \tag{2.11}$$

for $i = 1, 2$. Let $\ell(\cdot)$ be as defined in (2.3). Then the followings hold:

(i) *Problem (2.4) has a unique viscosity solution (θ, u) with the growth condition $\lim_{|z|\to\infty} u(z)/\ell(z) = 1$. Moreover, $0 \leq \theta \leq \overline{\theta}$, u is concave, and $z_i u_{z_i} \in C(\mathbb{R})$.*

(ii) *There are bounded functions $l_i^{\pm} : \mathbb{R} \mapsto \mathbb{R}$ and intervals $[b_i^{\pm}, s_i^{\pm}]$ satisfying*

$$\begin{aligned} (b_1^+, l_2^+(b_1^+)) &= (l_1^-(s_2^-), s_2^-), \quad & (l_1^+(s_2^+), s_2^+) &= (s_1^+, l_2^+(s_1^+)), \\ (l_1^-(b_2^-), b_2^-) &= (b_1^-, l_2^-(b_1^-)), \quad & (s_1^-, l_2(s_1^-)) &= (l_1^+(b_2^+), b_2^+), \end{aligned}$$

such that the whole region $\mathbb{R}^2$ is split into the following eight trading regions

$$\begin{array}{ll} \mathbf{SS} = [s_1^+, \infty) \times [s_2^+, \infty), & \mathbf{SN} = \{(z_1, z_2) \,|\, z_2 \in (b_2^+, s_2^+),\ z_1 \geqslant l_1^+(z_2)\}, \\ \mathbf{SB} = [s_1^-, \infty) \times (-\infty, b_2^+], & \mathbf{NB} = \{(z_1, z_2) \,|\, z_1 \in (b_1^-, s_1^-),\ z_2 \leqslant l_2^-(z_1)\}, \\ \mathbf{BB} = (-\infty, b_1^-] \times (-\infty, b_2^-], & \mathbf{BN} = \{(z_1, z_2) \,|\, z_2 \in (b_2^-, s_2^-),\ z_1 \leqslant l_1^-(z_2)\}, \\ \mathbf{BS} = (-\infty, b_1^+] \times [s_2^-, \infty), & \mathbf{NS} = \{(z_1, z_2) \,|\, z_1 \in (b_1^+, s_1^+),\ z_2 \geqslant l_2^+(z_1)\}, \end{array} \tag{2.12}$$

and one no-transaction region

$$\mathbf{NT} = \{(z_1, z_2) \,|\, l_1^-(z_2) < z_1 < l_1^+(z_2),\ l_2^-(z_1) < z_2 < l_2^+(z_1)\}. \tag{2.13}$$

Moreover, the boundary of each corner region **SS**, **SB**, **BS**, *and* **BB** *consists of one vertical and one horizontal half-line, whereas the boundary of each of* **SN**, **NS**, **BN**, *and* **NB** *consists of two parallel either vertical or horizontal half-lines and a curve in between connecting the endpoints of the two half-lines.*

The proof of Theorem 1 is deferred to Section 4. The uniqueness follows from a comparison principle (i.e., Lemma 4.1), whose proof is inspired by Hynd (2012) and Possamaï et al. (2015), and one of key steps is to introduce a suitable transformation such that the gradient constraint in (2.4) is transferred to a new one restricted in a closed and convex set including the origin. Regarding the existence, we consider an auxiliary problem (4.7), as is commonly used in ergodic control (see, e.g., Borkar (2006)). It is worth pointing out that to prove the existence, the existing literature (e.g., Hynd (2012); Possamaï et al. (2015)) typically adopts the standard Perron's argument by explicitly constructing appropriate sub and super solutions of problem (4.7). In contrast, we use the method introduced in Chen and Dai (2013) by first considering a related investment and consumption problem with a finite horizon and then sending the horizon to infinity. An advantage of this method is that many nice properties such as the concavity of the value function for the finite horizon problem are retained for the infinite horizon problem. Using the concavity and the growth condition, we can directly apply the argument in Chen and Dai (2013) to characterize the trading and no-transaction regions.

Remark 2.1. *Note that if (θ, u) is a solution of (2.4), then $(\theta, u + C)$ is also a solution of (2.4) for any constant C. However, this non-uniqueness of eigenfunctions has no impact on the eigenvalue. Therefore, the uniqueness of problem (2.4) means that all such eigenfunctions are considered identical.*

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Part (ii) of the above theorem indicates that the optimal trading policy is determined by the boundary of the no-transaction region, as depicted in Figure 1. In this paper, we shall further derive an asymptotic expansion with a small correlation ρ for the boundary of the no-transaction region as well as for the long-term excess growth rate θ.

To obtain the asymptotic expansion in an explicit form, we introduce new variables $\xi = (\xi_1, \xi_2)$ and function $U(\xi)$ by

$$\xi_i = \ln |z_i|, \quad U(\xi) = e^{-u(z) - \sum_i \beta_i \xi_i},$$

where (β_1, β_2) is the root of

$$\sum_{j=1}^{2} \sigma_{ij}\beta_j = r + \frac{\sigma_{ii}}{2} - \alpha_i \quad \forall\, i = 1, 2. \tag{2.14}$$

In a compact form, $\beta = \Sigma^{-1}(\mathbf{r} - \alpha + \frac{1}{2}\sigma_d)$ with $\sigma_d = (\sigma_{11}, \sigma_{22})^{tr}$.

Thanks to Theorem 1, the variational inequality (2.4) for (θ, u) can be transformed into the following free-boundary problem for (Θ, U) coupled with the unknown no-transaction region $\mathbf{NT}_\xi$ and the unknown optimal trading boundaries Γ_{ij} such that

$$\begin{cases} -\sigma_1^2 U_{\xi_1\xi_1} - \sigma_2^2 U_{\xi_2\xi_2} = \Theta U + 2\rho\sigma_1\sigma_2 U_{\xi_1\xi_2} & \text{in } \mathbf{NT}_\xi, \\ U_{\xi_i} + (\beta_i + k_{ij}z_i)U = 0 & \text{on } \Gamma_{ij}, \\ U_{\xi_i\xi_i} + [k_{ij}z_i - (\beta_i + k_{ij}z_i)^2]U = 0 & \text{on } \Gamma_{ij}, \end{cases} \tag{2.15}$$

where k_{ij} are defined as

$$k_{i1} = 1 + \lambda_i, \quad k_{i2} = 1 - \mu_i. \tag{2.16}$$

In addition, θ and Θ are related by the equation

$$\theta = \frac{1}{2}\Theta + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}\beta_i\beta_j. \tag{2.17}$$

And $\mathbf{NT}_\xi$ has a similar structure of $\mathbf{NT}$ stated in (2.13), i.e.,

$$\mathbf{NT}_\xi = \{(\xi_1, \xi_2)\,|\, l_{11}(\xi_2) < \xi_1 < l_{12}(\xi_2),\, l_{21}(\xi_1) < \xi_2 < l_{22}(\xi_1)\}.$$

where l_{ij} is the transform of $l_i^{\pm}$ in terms of the new variable ξ.

Note that $\beta = (\beta_1, \beta_2)$ defined in (2.14) depends on ρ and can be expanded in the following form:

$$\beta = \beta^0 + \rho\hat{\beta} + O(\rho^2), \quad \text{with } \beta^0 = \begin{pmatrix} \frac{r-\alpha_1}{\sigma_1^2} + \frac{1}{2} \\ \frac{r-\alpha_1}{\sigma_1^2} + \frac{1}{2} \end{pmatrix}, \quad \hat{\beta} = -\frac{1}{\sigma_1\sigma_2}\begin{pmatrix} r - \alpha_2 + \frac{1}{2}\sigma_2^2 \\ r - \alpha_1 + \frac{1}{2}\sigma_1^2 \end{pmatrix}. \tag{2.18}$$

So, based on the above notations and transformation, we seek the following formal asymptotic expansions in terms of small correlation ρ for (i) the long-term excess growth rate Θ, (ii) the optimal trading boundaries l_{ij}, and (iii) the eigenfunction U:

$$\Theta = \Theta^0 + \rho\hat{\Theta} + O(\rho^2), \tag{2.19}$$

$$l_{ij}(\breve{\xi}_i) = b_{ij} + \rho\,\hat{l}_{ij}(\breve{\xi}_j) + O(\rho^2), \quad \text{with } \breve{\xi}_1 := \xi_2,\ \breve{\xi}_2 := \xi_1, \tag{2.20}$$

$$U(\xi_1, \xi_2) = U^0(\xi_1, \xi_2) + \rho\,\hat{U}(\xi_1, \xi_2) + O(\rho^2), \tag{2.21}$$

where $\{\Theta^0, \hat{\Theta}, b_{ij}\}$ and $\{\hat{l}_{ij}, U^0, \hat{U}\}$ are respectively constants and functions given explicitly by the following theorem.

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Theorem 2. *Assume that* r, ρ, λ_i, μ_i, α_i, *and* σ_i *satisfy condition* (2.11). *Let* k_{ij} *for* $i, j = 1, 2$, β^0, *and* $\hat{\beta}$ *be defined by* (2.16), (2.18), *respectively. Then the formal asymptotic expansions proposed in* (2.19)-(2.21) *have an explicit form:*

(i) *Regarding the zeroth-order terms,*

$$\Theta^0 = \sigma_1^2\Theta_1 + \sigma_2^2\Theta_2, \tag{2.22}$$

$$b_{ij} = \ln\left|\frac{\gamma_{ij} - \beta_i^0}{k_{ij}}\right|, \tag{2.23}$$

$$U^0(\xi_1, \xi_2) = U_1(\xi_1)U_2(\xi_2), \tag{2.24}$$

where Θ_i *is a solution of the algebraic equation* (5.7), $\gamma_{ij} = \frac{1}{2} + \frac{(-1)^j}{2}\sqrt{1 - 4\beta_i^0 - 4\Theta_i}$, *and*

$$U_i(\xi_i) = \cos\left(\sqrt{\Theta_i}(\xi_i - b_{i1}) + \operatorname{arccot}\frac{\sqrt{\Theta_i}}{\gamma_{i1}}\right). \tag{2.25}$$

(ii) *Regarding the leading order terms,*

$$\hat{\Theta} = \frac{2\int_{b_{11}}^{b_{12}}\int_{b_{21}}^{b_{22}} U_1U_2[\sigma_1^2\hat{\beta}_1U_1'U_2 + \sigma_2^2\hat{\beta}_2U_1U_2' - \sigma_1\sigma_2U_1'U_2']d\xi_1d\xi_2}{\int_{b_{11}}^{b_{12}}\int_{b_{21}}^{b_{22}} U_1^2U_2^2d\xi_1d\xi_2}, \tag{2.26}$$

$$\hat{l}_{1j}(\xi_2) = \sum_{q=0}^{\infty}\frac{\psi_{2q}(\xi_2)}{U_2(\xi_2)}\sum_{p=1}^{\infty}c_{pq}\Theta_{1p}\frac{\psi_{1p}(b_{1j})}{H_{1j}}, \qquad \hat{l}_{1j}'(b_{2j}) = 0, \tag{2.27}$$

$$\hat{l}_{2j}(\xi_1) = \sum_{p=0}^{\infty}\frac{\psi_{1p}(\xi_1)}{U_1(\xi_1)}\sum_{q=1}^{\infty}c_{pq}\Theta_{2q}\frac{\psi_{2q}(b_{2j})}{H_{2j}}, \qquad \hat{l}_{2j}'(b_{1j}) = 0, \tag{2.28}$$

$$\hat{U}(\xi_1, \xi_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}c_{pq}\psi_{1p}(\xi_1)\psi_{2q}(\xi_2) - \sum_{i=1}^{2}\hat{\beta}_i\xi_i, \tag{2.29}$$

where $H_{ij} = [\gamma_{ij} - \beta_i^0][1 - 2\gamma_{ij}]U_i(b_{ij})$, c_{pq} *is given by* (5.11), Θ_{ip} *is the root of the algebraic equation* (5.10), *and* ψ_{ip} *is given explicitly by*

$$\psi_{ip}(\xi_i) = \cos\left(\sqrt{\Theta_i + \Theta_{ip}}(\xi_i - b_{i1}) + \operatorname{arccot}\frac{\sqrt{\Theta_i + \Theta_{ip}}}{\gamma_{i1}}\right). \tag{2.30}$$

Remark 2.2. (i) *Here we use the convention that if* $\Theta < 0$, *then* $\sqrt{\Theta} = \mathbf{i}\sqrt{-\Theta}$. *Also,*

$$\cos(\mathbf{i}x) = \cosh(x), \quad \cot(\mathbf{i}x) = -\mathbf{i}\coth x, \qquad \operatorname{arccot}(-\mathbf{i}\coth x) = \mathbf{i}x \quad \text{for } x \in \mathbb{R}.$$

(ii) *From Theorem 1, the boundary of each corner region* **SS**, **SB**, **BS**, *and* **BB** *consists of one vertical and one horizontal half-line. Thus, the fact that* $\hat{l}_{ij}'(b_{ij}) = 0$ *implies that the sell and buy boundaries of each asset are* C^1 *according to the leading order expansion. Graphically, the four trading boundaries are perpendicular to each other at every corner; see Figure 1.*

(iii) *In fact,* $\{\Theta_{ip}, \psi_{ip}\}_{p=0}^{\infty}$ *are all eigenpairs of the following eigenvalue problem*

$$\begin{cases} -\psi_{ip}'' - \Theta_i\psi_{ip} = \Theta_{ip}\psi_{ip} & \text{in } [b_{i1}, b_{i2}], \\ \psi_{ip}'(b_{ij}) + \gamma_{ij}\psi_{ip}(b_{ij}) = 0, & j = 1, 2, \\ \int_{b_{i1}}^{b_{i2}}\psi_{ip}^2(x)dx = 1. \end{cases} \tag{2.31}$$

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(iv) *The semi-closed form expansions relies on the linear structure of equation (2.15). In particular, if we "ignore" the dependence of β on ρ and treat β as an exogenous parameter, we can obtain a modified semi-closed form expansion by simply setting $\beta^0 = \beta$ and $\hat{\beta} = 0$ in the original expansion. As the modified expansion essentially comes from an approximation to the cross derivative term $2\rho\sigma_1\sigma_2 U_{x_1x_2}$ only, it should be more accurate than the original one, which is demonstrated by our numerical experiments (Figure 2). A detailed algorithm is presented in Section 3.2.*

Due to the heavy use of notations, we summarize the most important notations in Table 1 for comparison and later reference:

Table 1. Key Notations

	Long-Term Excess Growth Rate	Optimal Trading Boundaries	Eigenfunction
For Merton	$\overline{\theta}$	$\overline{\pi}$	NA
For the original variables z	θ	$l_i^{\pm}$	u
For the new variables ξ	Θ	l_{ij}	U
For the zeros order terms	Θ^0	b_{ij}	U^0
For the leading order terms	$\hat{\Theta}$	$\hat{l}_{ij}$	$\hat{U}$

Before giving the proof of the main results, we apply our formal asymptotic expansions and conduct an extensive numerical analysis.

3. Numerical Analysis

To test our asymptotic expansion, we first introduce the following finite difference method (FDM) to obtain a benchmark solution since a closed-form solution is absent.

3.1. Finite Difference Method

We use the penalty method presented in Dai and Zhong (2010) to solve the variational inequality (2.4) with an implicit finite difference scheme.[5] More precisely, we first consider the following approximation problem with a linear operator $\tilde{\mathcal{L}}$:

$$\min\left\{\tilde{\theta} - \tilde{\mathcal{L}}\tilde{u} - \frac{1}{2}\sigma_1^2[\tilde{u}_{\xi_1}^{(n)}]^2 - \frac{1}{2}\sigma_2^2[\tilde{u}_{\xi_2}^{(n)}]^2 - \rho\sigma_1\sigma_2\tilde{u}_{\xi_1}^{(n)}\tilde{u}_{\xi_2}^{(n)},\ \tilde{\mathcal{B}}[\tilde{u}],\ \tilde{\mathcal{S}}[\tilde{u}]\right\} = 0,$$

[5] Different from the policy iteration method used in Possamaï et al. (2015) and Altarovici et al. (2017), we first linearize the differential operator $\mathcal{A}$ in the variational inequality (2.4) by the Newton method, and then apply the non-smooth Newton method to handle the penalty approximation to the variational inequality.

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where $\xi_i = \ln|z_i|$, $(\tilde{\theta}, \tilde{u})$ is the unknown pair, and $(\tilde{\theta}^{(n)}, \tilde{u}^{(n)})$ is the (last) n-th iteration solution pair. Operators $\tilde{\mathcal{L}}$, $\tilde{\mathcal{B}}$, and $\tilde{\mathcal{S}}$ are defined, respectively, by

$$\begin{aligned}
\tilde{\mathcal{L}}\tilde{u} &= \frac{1}{2}\sigma_1^2\tilde{u}_{\xi_1\xi_1} + \rho\sigma_1\sigma_2\tilde{u}_{\xi_1\xi_2} + \frac{1}{2}\sigma_2^2\tilde{u}_{\xi_2\xi_2} + \Big(\alpha_1 - r - \frac{1}{2}\sigma_1^2 - \rho\sigma_1\sigma_2\tilde{u}_{\xi_2}^{(n)} - \sigma_1^2\tilde{u}_{\xi_1}^{(n)}\Big)\tilde{u}_{\xi_1} \\
&\quad + \Big(\alpha_2 - r - \frac{1}{2}\sigma_2^2 - \rho\sigma_1\sigma_2 u_{\xi_1}^{(n)} - \sigma_2^2 u_{\xi_2}^{(n)}\Big)u_{\xi_2}, \\
\tilde{\mathcal{B}}[\tilde{u}] &= \min\Big\{(1+\lambda_1)z_1 - \tilde{u}_{\xi_1},\ (1+\lambda_2)z_2 - \tilde{u}_{\xi_2}\Big\}, \\
\tilde{\mathcal{S}}[\tilde{u}] &= \min\Big\{-(1-\mu_1)z_1 + \tilde{u}_{\xi_1},\ -(1-\mu_2)z_2 + \tilde{u}_{\xi_2}\Big\}.
\end{aligned}$$

Note that, as mentioned in Remark 2.1, the eigenfunction u of the problem (2.4) is not unique. So, we will impose an artificial condition later.

Next, for a given parameter set, we restrict attention to a suitable bounded domain $D_\xi = (\underline{\xi}_1, \bar{\xi}_1) \times (\underline{\xi}_2, \bar{\xi}_2) \subset \mathbb{R}^2$ such that D_ξ contains the no-transaction region $\mathbf{NT}_\xi$. The numerical algorithm can be described as follows:

Step 1. Given n-th iteration result $(\tilde{\theta}^{(n)}, \tilde{u}^{(n)})$ with an initial guess (e.g., $(\tilde{\theta}^{(0)}, \tilde{u}^{(0)}) = (\overline{\theta}, \ell)$), solve the following linear problem for $(\tilde{\theta}, \tilde{u})$ in the domain D_ξ,

$$\begin{aligned}
0 &= \tilde{\theta} - \tilde{\mathcal{L}}\tilde{u} - \frac{1}{2}\sigma_1^2[\tilde{u}_{\xi_1}^{(n)}]^2 - \frac{1}{2}\sigma_2^2[\tilde{u}_{\xi_2}^{(n)}]^2 - \rho\sigma_1\sigma_2\tilde{u}_{\xi_1}^{(n)}\tilde{u}_{\xi_2}^{(n)} \\
&\quad + P \times \sum_{i=1}^{2} \mathbf{1}_{\{(1+\lambda_i)z_i - \tilde{u}_{\xi_i}^{(n)} < 0\}}\Big((1+\lambda_i)z_i - \tilde{u}_{\xi_i}\Big) \\
&\quad + P \times \sum_{i=1}^{2} \mathbf{1}_{\{\tilde{u}_{\xi_i}^{(n)} - (1-\mu_i)z_i < 0\}}\Big(\tilde{u}_{\xi_i} - (1-\mu_i)z_i\Big),
\end{aligned}$$

coupled with boundary conditions: $u(\underline{\xi}_1, \underline{\xi}_2) = 1$, and

$$\begin{aligned}
\tilde{u}_{\xi_1}(\underline{\xi}_1, \xi_2) &= (1+\lambda_1)\underline{z}_1, \quad \tilde{u}_{\xi_2}(\xi_1, \underline{\xi}_2) = (1+\lambda_2)\underline{z}_2, \\
\tilde{u}_{\xi_1}(\bar{\xi}_1, \xi_2) &= (1-\mu_1)\bar{z}_1, \quad \tilde{u}_{\xi_2}(\xi_1, \bar{\xi}_2) = (1-\mu_2)\bar{z}_2.
\end{aligned}$$

Here $\mathbf{1}_{\{\cdot\}}$ is an indicator function, and P is a big positive constant (e.g., $P = 10^5$) that is known as the penalty parameter.

Step 2. Calculate the relative error

$$RE = \frac{||(\tilde{\theta}, \tilde{u}) - (\tilde{\theta}^{(n)}, \tilde{u}^{(n)})||}{||(\tilde{\theta}^{(n)}, \tilde{u}^{(n)})||}.$$

If the relative error $RE < \epsilon$, where ϵ is a given small constant (e.g., $\epsilon = 10^{-12}$), then set $(\theta, u) = (\tilde{\theta}, \tilde{u})$ and stop; Otherwise set $(\tilde{\theta}^{(n+1)}, \tilde{u}^{(n+1)}) = (\tilde{\theta}, \tilde{u})$ and go to step 1.

In our numerical experiment, we regard the solution with the mesh size of 800×800 as the benchmark; see the dotted line in Figure 1. The basic parameter values are summarized in Table 2. That is, the interest rate is 3% per year. For risky asset one, the expected return rate is 10% and the volatility is 20%. For risky asset two, the expected return rate and volatility are 12% and 25%, respectively. Finally, the proportion of transaction cost of each stock is set to be 1% for both selling and buying. This computation is relatively costly, and it takes about 15 minutes to get a solution by a normal laptop.

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Table 2. Basic Parameters

	Risk-free Rate	Risky Asset One				Risky Asset Two			
Notation	r	α_1	σ_1	λ_1	μ_1	α_2	σ_2	λ_2	μ_2
Value (%)	3	10	20	1	1	12	25	1	1

3.2. **Asymptotic Method**

Thanks to Theorem 2, one can directly apply the expansion. Note that in our formal expansion, the leading order terms are expressed in a series form. By a rule of thumb, we take the first 10 terms in our numerical algorithm, which is described below:

Step 1. Solve the algebraic equation (5.7) to obtain Θ_i;

Step 2. Use Θ_i to get b_{ij} and U_i from (2.23) and (2.25), respectively;

Step 3. Solve for $\hat{\Theta}$ by the integration formula (2.26);

Step 4. For $p = 1, 2, \cdots, 10$, solve the algebraic equation (5.10) to obtain Θ_{ip}, and then calculate eigenfunction ψ_{ip} by (2.30);

Step 5. For $p, q = 0, 1, 2, \cdots, 10$, solve for coefficients c_{pq}, trading boundaries $\hat{l}_{ij}$, and eigenfunction $\hat{U}$ by (5.11), (2.27), (2.28), and (2.29), respectively;

Step 6. Calculate the long-term growth rate $\Theta = \sigma_1^2\Theta_1 + \sigma_2^2\Theta_2 + \rho\hat{\Theta}$ and the optimal trading boundary $l_{ij} = b_{ij} + \rho\,\hat{l}_{ij}$;

Step 7. Finally, calculate the long-term excess growth rate by (2.17), i.e., $\theta = \frac{1}{2}\Theta + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_{ij}\beta_i\beta_j$, and the optimal trading boundary $l_i^{\pm}$ is derived from l_{ij} by a change of variable $\xi_i = \ln|z_i|$.

As discussed in Remark 2.2 (iv), we can obtain a modified expansion by simply setting $\beta^0 = \beta$ and $\hat{\beta} = 0$ in the above algorithm. It turns out that this modified expansion not only has a more compact form as the terms related to $\hat{\beta}$ disappear, but also performs more accurately even for a relatively large correlation as explained in Remark 2.2 (iv). Figure 2 gives a comparison of the two expansions. When correlation is relatively small ($\rho = 0.01$, left panel), the no-transaction regions generated by both the original expansion (solid blue line) and the modified expansion (solid gray line) are very close to the benchmark (dotted red line). When correlation is relatively large ($\rho = 0.1$, right panel), the no-transaction region generated by the original expansion (solid blue line) deviates from the benchmark (dotted red line) in a notable size. However, the modified expansion (solid gray line) still works very well. Hence, we will use the modified expansion in subsequent numerical analysis.

Since our expansion admits an explicit form, it is obvious that our asymptotic method is much more efficient than FDM. In fact, for a normal laptop, it costs several seconds to obtain a solution by our asymptotic method. By contrast, the FDM usually takes about 15 minutes to get a solution and may be unstable when transaction costs are tiny.

3.3. **Optimal Trading Boundaries**

Let us consider the optimal trading boundaries. Figures 1, 3, and 4 illustrate some numerical experiments on optimal trading boundaries. More specifically, in each plot, the thick black (resp. blue) dot is the Merton's strategy in the absence of transaction costs but with a non-zero (resp. zero) correlation; The blue dashed rectangle is the uncorrelated case (i.e., $\rho = 0$) but with transaction costs; The dotted and solid lines are the optimal trading boundaries obtained

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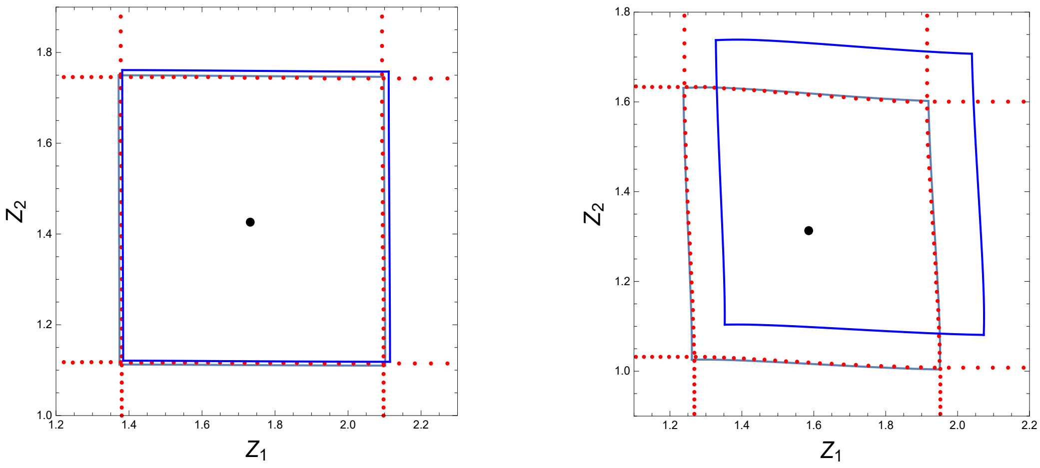

Figure 2. Comparison between Original Expansion and Modified Expansion. In each plot, the thick black dot is the Merton's strategy in the absence of transaction costs but with a non-zero correlation. Default parameters are given by Table 2. *Left Panel*: When correlation is relatively small $\rho = 0.01$, the no-transaction regions obtained by both the original expansion (solid blue line) and the modified expansion (solid gray line) are close to the benchmark (dotted red line). *Right Panel*: When correlation is relatively large $\rho = 0.1$, the no-transaction region by the original expansion (solid blue line) deviates from the benchmark (dotted red line) in a notable size. However, the modified expansion method (solid gray line) still works very well.

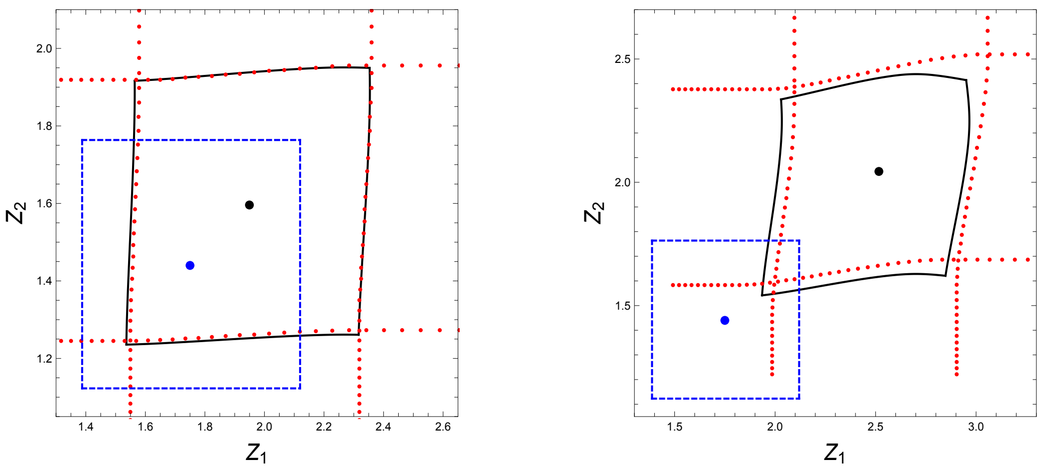

Figure 3. The No-transaction Regions with Different Correlations (Left: $\rho = -0.1$, Right: $\rho = -0.3$). In each plot, the thick black (resp. blue) dot is the Merton's strategy in the absence of transaction costs but with a non-zero (resp. zero) correlation; The blue dashed rectangle is the uncorrelated case (i.e., $\rho = 0$) but with transaction costs; The red dotted lines and the black solid lines correspond to the benchmark and the asymptotic expansion, respectively, in the presence of both correlation and transaction costs. Other parameters are given in Table 2.

by the FDM and the asymptotic expansion, respectively, in the presence of both correlation and transaction costs. Default parameters are summarized in Table 2. As you can see, for $|\rho| = 0.1$ (see left panels in both Figures 1 and 3), our asymptotic expansion has an impressive accuracy compared with the benchmark result by FDM. For an even larger correlation $|\rho| = 0.3$ (see right panels in both Figure 1 and 3), our asymptotic expansion also provides a good approximation.

Moreover, since our expansion is for a small correlation ρ and does not rely on the assumption of small transaction costs, they should work equally well for different levels of transaction costs provided that the correlation is relatively small. Figure 4 compares our expansion (black solid

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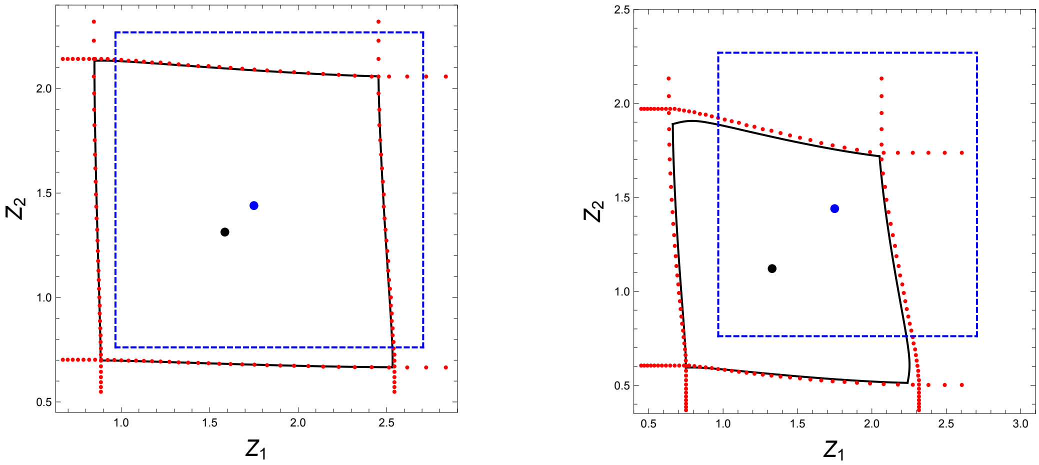

Figure 4. No-transaction Regions for Large Transaction Costs ($\lambda_i = \mu_i = 10\%$). In each plot, the thick black (resp. blue) dot is the Merton's strategy in the absence of transaction costs but with a non-zero (resp. zero) correlation; The blue dashed rectangle is the uncorrelated case (i.e., $\rho = 0$) but with transaction costs; The red dotted lines and black solid lines correspond to the benchmark and our expansion, respectively, in the presence of both correlation and transaction costs. Other parameters are summarized in Table 2.

lines) with the FDM (red dotted lines) for a large transaction cost level (i.e., $\mu_i = \lambda_i = 10\%$ for $i = 1, 2$). The left panel shows that, as with small transaction costs, our expansion performs impressively well when the correlation level is 0.1. Even for a much larger correlation, say 0.3, the right panel of Figure 4 indicates that our expansion still provides a good approximation. More interestingly, by comparing the right panels in Figures 1 and 4, we find that facing a large correlation, the trading boundaries (black solid lines) with a large transaction cost match the benchmark (red dotted lines) better than the case with a small transaction cost. This suggests that our modified asymptotic expansion method can be potentially applied to investments with very illiquid assets such as housing, art products, and so on.

Next, we turn to the impact of correlation on the optimal trading boundaries. Compared to the uncorrelated case, the no-transaction region is no longer a rectangle but a quadrangle with curved boundaries being orthogonal to each other at four corners. In the presence of correlation, the two stocks have substitution effects each other, which leads the rectangle to be a quadrangle. However, the boundaries are perpendicular to each other at the corner, as verified by the fact that $\hat{l}'_{ij}(b_{ij}) = 0$ for the leading order expansion. From Theorem 1, the boundary of each corner region **SS**, **SB**, **BS**, and **BB** consists of one vertical and one horizontal half-line. Thus, this orthogonal property at four corners implies that the sell and buy boundaries of each asset are C^1 for the leading order expansion.

Figures 1 and 3 further show two interesting features: (a) The no-transaction regions shift along the 45-degree line in the two-asset plane upward and downward for negative correlations and positive correlations, respectively. (b) The slopes of optimal boundaries tend to be positive (negative) for negative (positive) correlations. These can be interpreted from the diversification effect. Indeed, a more negative correlation implies a bigger diversification benefit, thus, investors should invest more in risky assets. This leads to the shift of the no-transaction region upward along the 45-degree line in the two-asset plane. Meanwhile, when the holdings in one risky asset increase, investors should allocate a larger position in the other risky asset for hedging purposes since they are negatively correlated. Hence, the slopes of optimal boundaries are positive in the case of negative correlation. Similar results can be obtained in the case of positive correlation.

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Finally, correlations seem to have no notable impact on the width of the no-transaction region that is significantly affected by transaction costs.

3.4. **Long-Term Excess Growth Rate**

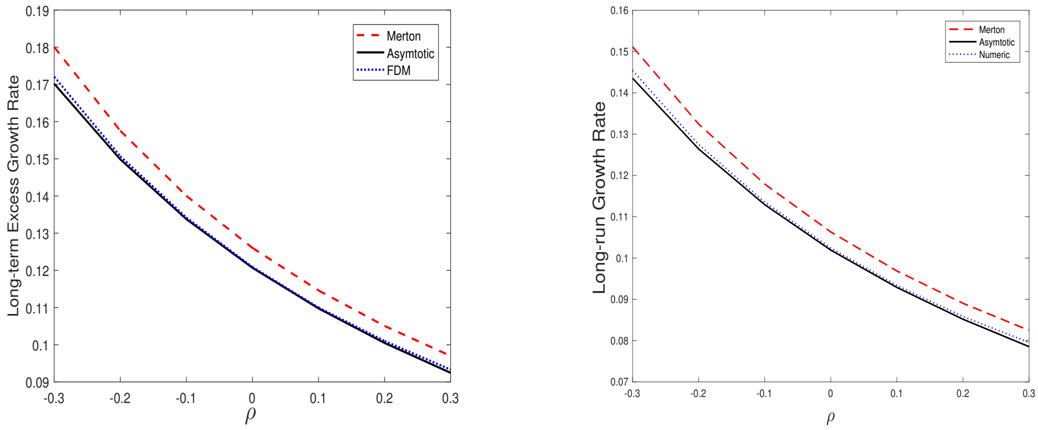

Figure 5. Long-term Excess Growth Rate. In each plot, the red dashed, blue dotted, and black solid lines represent the long-term excess growth rates obtained by Merton's solution (i.e., $\overline{\theta}$), the FDM (i.e., θ_f), and our expansion (i.e., θ_a), respectively. *Left Panel*: Basic parameter values are summarized in Table 2. *Right Panel*: Empirical parameter values from Flavin and Yamashita (2002) for housing and stock markets. That is, the risk-free rate $r = 1\%$; for the relatively liquid risky asset (stock): $\alpha_1 = 8\%$, $\sigma_1 = 24\%$, $\lambda_1 = \mu_1 = 0.1\%$; for the illiquid risky asset (house) $\alpha_2 = 6\%$, $\sigma_2 = 14\%$, $\lambda_2 = \mu_2 = 3\%$.

In this subsection, we investigate the eigenvalue θ, which stands for the portfolio's optimal long-term excess growth rate.

First, to measure the accuracy of our expansion, one way is to simply compare the long-term excess growth rates obtained by the FDM and by our modified asymptotic expansion, which are denoted by θ_f and θ_a, respectively. Figure 5 gives such a direct comparison. In each plot, the red dashed, blue dotted, and black solid lines represent the long-term excess growth rates obtained by Merton's solution (i.e., $\overline{\theta}$), the FDM (i.e., θ_f), and our expansion (i.e., θ_a), respectively. The left panel shows the results with our basic parameter values in Table 2, while the right panel present the results for the parameter values reported in Flavin and Yamashita (2002) studying the housing and stock markets.

For both two sets of parameter values, we can see that the long-term excess growth rates obtained by our expansion (black solid line) and by the FDM (blue dotted line) are pretty close even for a large correlation $|\rho| = 0.3$.[6] Of course, all these long-term excess growth rates are bounded from above by that in Merton's economy (red dashed line). In addition, there is an interesting observation that the long-term excess growth rate obtained by our expansion is always less than that obtained by the FDM, which implies that the second order term in our expansion (i.e., the term $O(\rho^2)$) may contribute positively to the long-term investment. The difference tends to be large as the magnitude of correlation increases but in a non-symmetric way. That is, for the same magnitude of correlation, the difference with a positive correlation is less than that with a negative correlation. This suggests that our expansion performs better in the case of a positive correlation. Moreover, as the correlation increases from negative to

[6] In fact, we find that the difference is typically less than 0.3% for $|\rho| \leq 0.3$.

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positive, the long-term excess growth rate is monotonically decreasing for both cases with and without transaction costs. This is because the effect of diversification becomes stronger as the correlation tends to be -1.

Table 3. Comparison of Long-term Excess Growth Rates. The row with zero transaction costs (i.e., $\mu_i = \lambda_i = 0\%$) displays the long-term excess growth rates of Merton's solution (i.e., $\overline{\theta}$) for different correlations. For other transaction costs ranging from 0.1% to 10%, each has three rows of data: The upper and middle rows collect the long-term excess growth rates calculated by the benchmark FDM (i.e., θ_f) and our expansion (i.e., θ_a), respectively. The lower row shows the implied growth rates that are associated with implementing the trading strategy from our expansion (i.e., $\tilde{\theta}_a$). Other parameter values are documented in Table 2.

Costs	**Correlation ρ**							
$\mu_i = \lambda_i$	**-0.3**	**-0.2**	**-0.1**	**0**	**0.1**	**0.2**	**0.3**	
0%	18.01%	15.76%	14.01%	12.61%	11.46%	10.51%	9.70%	$\overline{\theta}$
0.1%	17.94%	15.70%	13.96%	12.56%	11.43%	10.47%	9.68%	θ_f
	17.66%	15.53%	13.85%	12.48%	11.34%	10.37%	9.53%	θ_a
	17.86%	15.64%	13.91%	12.52%	11.39%	10.44%	9.64%	$\tilde{\theta}_a$
0.2%	17.83%	15.60%	13.87%	12.50%	11.36%	10.42%	9.62%	θ_f
	17.55%	15.44%	13.77%	12.42%	11.28%	10.32%	9.48%	θ_a
	17.76%	15.55%	13.83%	12.46%	11.33%	10.39%	9.58%	$\tilde{\theta}_a$
0.3%	17.73%	15.52%	13.80%	12.43%	11.30%	10.36%	9.57%	θ_f
	17.46%	15.37%	13.71%	12.36%	11.23%	10.27%	9.44%	θ_a
	17.67%	15.47%	13.77%	12.40%	11.28%	10.34%	9.55%	$\tilde{\theta}_a$
0.4%	17.64%	15.44%	13.73%	12.37%	11.25%	10.32%	9.53%	θ_f
	17.38%	15.30%	13.65%	12.31%	11.19%	10.23%	9.41%	θ_a
	17.59%	15.40%	13.70%	12.34%	11.23%	10.30%	9.51%	$\tilde{\theta}_a$
0.5%	17.56%	15.37%	13.67%	12.32%	11.20%	10.28%	9.49%	θ_f
	17.31%	15.24%	13.60%	12.26%	11.15%	10.20%	9.37%	θ_a
	17.51%	15.34%	13.64%	12.29%	11.18%	10.26%	9.47%	$\tilde{\theta}_a$
1%	17.26%	15.12%	13.45%	12.12%	11.03%	10.12%	9.35%	θ_f
	17.02%	14.99%	13.38%	12.07%	10.98%	10.05%	9.24%	θ_a
	17.23%	15.09%	13.44%	12.11%	11.02%	10.11%	9.33%	$\tilde{\theta}_a$
5%	15.89%	13.94%	12.43%	11.23%	10.24%	9.41%	8.71%	θ_f
	15.68%	13.83%	12.37%	11.19%	10.20%	9.36%	8.63%	θ_a
	15.87%	13.92%	12.41%	11.21%	10.23%	9.40%	8.69%	$\tilde{\theta}_a$
10%	14.92%	13.10%	11.70%	10.57%	9.65%	8.88%	8.23%	θ_f
	14.70%	12.99%	11.63%	10.53%	9.61%	8.84%	8.17%	θ_a
	14.89%	13.08%	11.68%	10.56%	9.64%	8.87%	8.21%	$\tilde{\theta}_a$

We now implement the trading boundaries obtained by the expansion to calculate the implied long-term excess growth rate, denoted by $\tilde{\theta}_a$, which can be obtained by solving the equation in the no-transaction region with the trading condition on the *known* boundaries from the expansion. That is, $(\tilde{\theta}_a, \hat{u})$ solves the following eigenvalue problem

$$\begin{cases} \tilde{\theta}_a - \mathcal{A}[\tilde{u}] = 0 & \text{in } \tilde{\mathbf{NT}}_a, \\ \tilde{u}_{z_i} = 1 + \lambda_i & \text{on } \tilde{l}_i^- \quad \text{for } i = 1, 2, \\ \tilde{u}_{z_i} = 1 - \mu_i & \text{on } \tilde{l}_i^+ \quad \text{for } i = 1, 2, \end{cases} \tag{3.1}$$

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where the operator $\mathcal{A}$ is defined in (2.5), $\tilde{l}_i^{\pm}$ are optimal trading boundaries obtained from the expansion, and the approximated no-transaction region is given by

$$\tilde{\mathbf{NT}}_a = \{(z_1, z_2) \,|\, \tilde{l}_1^-(z_2) < z_1 < \tilde{l}_1^+(z_2),\, \tilde{l}_2^-(z_1) < z_2 < \tilde{l}_2^+(z_1)\}.$$

One can use a finite difference scheme to numerically solve (3.1).

Table 3 provides a comparison of the long-term excess growth rates obtained by the three different methods with different levels of correlations and transaction costs. The row with zero transaction costs (i.e., $\mu_i = \lambda_i = 0\%$) displays the long-term excess growth rates of Merton's solution (i.e., $\overline{\theta}$) for different correlations. For other transaction cost levels ranging from 0.1% to 10%, each has three rows of data in a group: The upper and middle rows collect the long-term excess growth rates calculated by the benchmark FDM (i.e., θ_f) and our expansion (i.e., θ_a), respectively. The lower row shows the implied growth rates that are associated with implementing the trading strategy from our expansion (i.e., $\tilde{\theta}_a$). Other parameter values are given in Table 2. As one can see, for each fixed pair of correlation and transaction costs, both $\tilde{\theta}_a$ and θ_a are less than the benchmark θ_f, but the former is closer to the benchmark than the latter. Overall, the absolute error is less than 0.3%. For each fixed transaction cost level (each row in Table 3), consistent with the observation in Figure 5, negative correlation increases the diversification effect and thus leads to a higher long-term excess growth rate. Meanwhile, for each fixed correlation level (i.e., each column in Table 3), as transaction costs decrease, the spread of long-term excess growth rates calculated by $\theta_a - \theta_f$ or $\tilde{\theta}_a - \theta_f$ tends to increase. However, this is by no means to say that our expansion is less accurate for small transaction costs. In fact, a small transaction cost leads to a small no-transaction region. As a result, the penalty method, which relies on the equation in the no-transaction region, becomes less stable. In particular, we find that both the solvency domain and the penalty constant must be carefully chosen to guarantee the convergence of the numerical scheme. By contrast, transaction costs have no impact on our expansion due to its closed-form.

4. Proof of Theorem 1

To prove Theorem 1, we begin with the uniqueness of the eigenvalue problem (2.4). For later use, we define the following closed convex subset of $\mathbb{R}^2$ and corresponding support function

$$E := \{z \in \mathbb{R}^2 : -\lambda_i \le z_i \le \mu_i,\ i = 1, 2\}, \quad \ell_E(z) = \sup_{x \in E} x \cdot z \quad \text{for } z \in \mathbb{R}^2. \tag{4.1}$$

4.1. Uniqueness

The uniqueness follows immediately from the following comparison principle.

Lemma 4.1 (Comparison Principle). *Suppose that u_1 is a viscosity subsolution of (2.4) with eigenvalue θ_1 and that u_2 is a viscosity supersolution of (2.4) with eigenvalue θ_2. Assume further that*

$$\limsup_{|z|\to\infty} \frac{u_1(z)}{\ell(z)} \le 1 \le \liminf_{|z|\to\infty} \frac{u_2(z)}{\ell(z)}.$$

Then, we have $\theta_1 \le \theta_2$.

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Proof. First, let C be a positive constant to be specified later, and define

$$\tilde{\theta}_1 = -\theta_2 + C, \quad \tilde{u}_1(z) = -u_2(z) + z_1 + z_2, \tag{4.2}$$

$$\tilde{\theta}_2 = -\theta_1 + C, \quad \tilde{u}_2(z) = -u_1(z) + z_1 + z_2. \tag{4.3}$$

Since u_1 (resp. u_2) is a viscosity subsolution (resp. supersolution) of (4.2) associated with the eigenvalue θ_1 (resp. θ_2), one can easily check that $\tilde{u}_1$ (resp. $\tilde{u}_2$) is a viscosity subsolution (resp. supersolution) of the following eigen problem associated with the eigenvalue $\tilde{\theta}_1$ (resp. $\tilde{\theta}_2$):

$$\max\left\{\tilde{\theta} - \frac{1}{2}Tr[\Sigma_z D^2\tilde{u}] - g(D\tilde{u}) - f(z),\ \tilde{\mathcal{B}}[\tilde{u}],\ \tilde{\mathcal{S}}[\tilde{u}]\right\} = 0 \quad \text{in } \mathbb{R}^2, \tag{4.4}$$

where $Tr[\cdot]$ is the trace operator,

$$\begin{aligned}
\Sigma_z &= \begin{pmatrix} \sigma_1^2 z_1^2 & \rho\sigma_1\sigma_2 z_1 z_2 \\ \rho\sigma_1\sigma_2 z_1 z_2 & \sigma_2^2 z_2^2 \end{pmatrix}, \qquad (4.5)\\
g(D\tilde{u}) &= \frac{1}{2}\sum_i^2\sum_j^2 \sigma_{ij} z_i z_j[\tilde{u}_{z_i}\tilde{u}_{z_j} - \tilde{u}_{z_i} - \tilde{u}_{z_j}] + \sum_i^2(\alpha_i - r)z_i\tilde{u}_{z_i},\\
f(z) &= \frac{1}{2}\sum_i^2\sum_j^2 \sigma_{ij} z_i z_j - \sum_i^2(\alpha_i - r)z_i + C,\\
\tilde{\mathcal{B}}[\tilde{u}] &= \max\{\tilde{u}_{z_1} - \mu_1, \tilde{u}_{z_2} - \mu_2\},\\
\tilde{\mathcal{S}}[\tilde{u}] &= \max\{-\tilde{u}_{z_1} + \lambda_1, -\tilde{u}_{z_2} + \lambda_2\}.
\end{aligned}$$

In addition, from the growth condition for u_1 and u_2, we have the following growth condition for $\tilde{u}_1$ and $\tilde{u}_2$:

$$\limsup_{|z|\to\infty}\frac{\tilde{u}_1(z)}{\ell_E(z)} \le 1 \le \liminf_{|z|\to\infty}\frac{\tilde{u}_2(z)}{\ell_E(z)}.$$

We choose C sufficiently large such that the convex function f is non-negative. Next, we will apply a similar argument as in the proof of Theorem 3.1 in Possamaï et al. (2015) to obtain a comparison principle for problem (4.2) with the above growth condition, i.e., $\tilde{\theta}_1 \le \tilde{\theta}_2$.

From the definition of the function $\ell_E(\cdot)$ in (2.3), there exist $L, L' > 0$ such that

$$L'|z| \le \ell_E(z) \le L|z|. \tag{4.6}$$

Next, fix some $\eta > 0$, $0 < \tau < 1$, and define, by de-doubling variables technique,

$$\overline{u}(x,z) := \tau\tilde{u}_1(x) - \tilde{u}_2(z) - \frac{1}{2\eta}|x-z|^2 \quad \text{for } (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Since $\tilde{u}_1$ is a viscosity subsolution of (2.4), we have $D\tilde{u}_1 \in E$ in the viscosity solution sense. Then, for $z \neq 0$, we have

$$\begin{aligned}
\overline{u}(x,z) &= \tau\Big(\tilde{u}_1(x) - \tilde{u}_1(z)\Big) + \Big(\tau\tilde{u}_1(z) - \tilde{u}_2(z)\Big) - \frac{1}{2\eta}|x-z|^2\\
&\le \Big(\tau\tilde{u}_1(z) - \tilde{u}_2(z)\Big) + \tau L|x-z| - \frac{1}{2\eta}|x-z|^2\\
&= \ell_E(x)\left(\tau\frac{\tilde{u}_1(z)}{\ell_E(x)} - \frac{\tilde{u}_2(z)}{\ell_E(z)}\right) + \tau L|x-z| - \frac{1}{2\eta}|x-z|^2.
\end{aligned}$$

By the growth conditions on u_1 and u_2 and condition (4.6), this implies that

$$\lim_{|(x,z)|\to\infty}\overline{u}(x,z) = -\infty.$$

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Then, $\overline{u}(x,z)$ has a global maximizer $(x^{\tau,\eta}, z^{\tau,\eta})$. So, by the Crandall-Ishii Lemma (see Theorem 3.2 in Crandall et al. (1992)), we deduce that for any $\eta > 0$, there exist symmetric positive matrices X and Y such that

$$\begin{aligned} \left(\frac{1}{\eta}(x^{\tau,\eta} - z^{\tau,\eta}), X\right) &\in \bar{J}^{2,+}(\tau u_1)(x^{\tau,\eta}), \\ \left(\frac{1}{\eta}(x^{\tau,\eta} - z^{\tau,\eta}), Y\right) &\in \bar{J}^{2,-}(u_2)(z^{\tau,\eta}), \end{aligned}$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \eta A^2, \quad \text{with } A = \frac{1}{\eta}\begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix},$$

which implies that $X \leq Y$.

Since $\tilde{u}_1$ is a viscosity subsolution, we have $D\tilde{u}_1 \in E$ in viscosity solution sense. Thus,

$$-\lambda_i \leq \frac{1}{\tau\eta}(x_i^{\tau,\eta} - z_i^{\tau,\eta}) \leq \mu_i, \quad \text{for } i = 1, 2.$$

which in turn implies, by using $\tau \in (0,1)$, that

$$-\lambda_i < \frac{1}{\eta}(x_i^{\tau,\eta} - z_i^{\tau,\eta}) < \mu_i, \quad \text{for } i = 1, 2.$$

Thanks to the above strictly inequality and the fact that $\tilde{u}_1$ (resp. $\tilde{u}_2$) is a viscosity subsolution (resp. supersolution) of (2.4), we immediately have

$$\begin{aligned} \tilde{\theta}_1 - \frac{1}{2\tau}Tr(\Sigma_z X) - g\left(\frac{1}{\tau\eta}(x^{\tau,\eta} - z^{\tau,\eta})\right) - f(x^{\tau,\eta}) &\leq 0, \\ \tilde{\theta}_2 - \frac{1}{2}Tr(\Sigma_z Y) - g\left(\frac{1}{\eta}(x^{\tau,\eta} - z^{\tau,\eta})\right) - f(z^{\tau,\eta}) &\geq 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \tau\tilde{\theta}_1 - \tilde{\theta}_2 &\leq \frac{1}{2}Tr(\Sigma_z(X-Y)) + \tau g\left(\frac{x^{\tau,\eta} - z^{\tau,\eta}}{\tau\eta}\right) - g\left(\frac{x^{\tau,\eta} - z^{\tau,\eta}}{\eta}\right) + \tau f(x^{\tau,\eta}) - f(z^{\tau,\eta}) \\ &\leq \tau g\left(\frac{x^{\tau,\eta} - z^{\tau,\eta}}{\tau\eta}\right) - g\left(\frac{x^{\tau,\eta} - z^{\tau,\eta}}{\eta}\right) + f(x^{\tau,\eta}) - f(z^{\tau,\eta}). \end{aligned}$$

By standard techniques from the theory of viscosity solutions, we then construct a $z^\tau \in \mathbb{R}^2$ and a sequence $(\eta_n)_{n\geq 0}$ converging to zero such that $(x^{\tau,\eta_n}, z^{\tau,\eta_n}) \longrightarrow (z^\tau, z^\tau)$ as $n \to \infty$. Passing to the limit in the above inequality and using the fact $g(0) = 0$, we have $\tau\tilde{\theta}_1 - \tilde{\theta}_2 \leq 0$, leading to $\tilde{\theta}_1 \leq \tilde{\theta}_2$ due the arbitrariness of $\tau \in (0,1)$.

Finally, $\theta_1 \leq \theta_2$ follows immediately from the definition of $\tilde{\theta}_i$ for $i = 1, 2$. □

4.2. **Existence**

Now let us turn to the existence of a solution of problem (2.4). Motivated by the method used in ergodic control (see, e.g., Borkar (2006); Hynd (2012); Possamaï et al. (2015)), we consider the following auxiliary problem, for any $\delta > 0$,

$$\min\{\delta u_\delta - \mathcal{A}[u_\delta],\ \mathcal{B}[u_\delta],\ \mathcal{S}[u_\delta]\} = 0 \qquad \text{in } \mathbb{R}^2, \tag{4.7}$$

where u_δ is the unknown, and operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$ are the same as in the problem (2.4).

Lemma 4.2. *Problem (4.7) has a unique viscosity solution with the growth condition* $\lim_{|z|\to\infty} u_\delta(z)/\ell(z) = 1$. *Moreover,*

(i) u_δ *is Lipschitz continuous and concave;*

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(ii) *The following estimate holds:*

$$0 \le u_\delta(\cdot) - \ell(\cdot) \le \frac{\overline{\theta}_\delta}{\delta}, \tag{4.8}$$

where $\overline{\theta}_\delta$ *is given in* (2.9) *with* $\alpha_i - r$ *replaced by* $\alpha_i - r - \delta$.

Proof. The uniqueness can be obtained from a comparison principle similar to Lemma 4.1, whose proof is therefore omitted.

To prove the existence, different from the existing literature (e.g., Hynd (2012) and Possamaï et al. (2015)) in which Perron's argument by explicitly is used to construct appropriate sub and super solutions of problem (4.7), we will use the method introduced in Chen and Dai (2013) by first considering a related investment and consumption problem with finite horizon $T < \infty$. Then, the solution of (4.7) is obtained as the limit of the solution for the finite horizon problem when T goes to infinity.

To be more precise, consider the following investment and consumption problem with finite horizon T:

$$\sup_{(L,M,C)\in\mathcal{C}_c(x,y)} \mathbb{E}_t^{x,y}\Big[-\int_t^T \kappa\, e^{-\nu C_s} ds - e^{\nu(X_T+\ell(Y_T))}\Big],$$

where $\kappa > 0$ is a weight of consumption rate C_t, $\mathcal{A}_c(x,y)$ is the set of all admissible strategies, and X_t and Y_t satisfy dynamics (2.1) with r and α_i being replaced by δ and $\alpha_i - r$, respectively. Then, the lemma follows immediately from Theorems 2.1 and 2.2 in Chen and Dai (2013). ☐

Proof of Theorem 1 (i). It remains to prove the existence. Since u_δ is Lipschitz and the estimate (4.8) holds, we can use a diagonalization argument to obtain the following convergence: There exists a sequence $\delta_k > 0$ tending to 0 as $k \to \infty$, such that

$$\lim_{k\to\infty} \delta_k u_{\delta_k} = \theta \in \mathbb{R}, \quad \text{and} \quad u_{\delta_k} \longrightarrow u \in C(\mathbb{R}^2) \text{ locally uniform as } k \to \infty.$$

Then, the assertion that u is a concave viscosity solution of (2.4) associated with the eigenvalue θ, follows directly from the definition of viscosity solutions by passing to the limit under local uniform convergence. In addition, the uniqueness comes directly from the comparison principle.

Note that $\overline{\theta}_\delta \to \overline{\theta}$ as $\delta \searrow 0$, thus (4.8) implies that $0 \le \theta \le \overline{\theta}$. Finally, the regularity result $z_i u_{z_i} \in C(\mathbb{R}^2)$ follows from Theorem 2.3 in Chen and Dai (2013). This completes the proof of Theorem 1 (i). ☐

4.3. **No-Transaction Region**

Our existence proof suggests that u is Lipschitz concave and satisfies the growth condition. This allows us to follow a similar argument as in Chen and Dai (2013) to prove the rest part of Theorem 1.

Proof of Theorem 1 (ii). Let us first decompose the operator $\mathcal{A}[u]$ as $\mathcal{A}[u] = \frac{1}{2}Tr[\Sigma_z D^2 u] - f(z)$, where Σ_z is defined in (4.5), and

$$f(z) = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}(z_i u_{z_i} - \overline{\pi}_i)(z_j u_{z_j} - \overline{\pi}_j) + \theta - \overline{\theta}$$

with $(\overline{\pi}_1, \overline{\pi}_2)$ and $\overline{\theta}$ being respectively the Merton's optimal portfolio allocation and long-term excess return rate given in (2.10) and (2.9), i.e.,

$$(\overline{\pi}_1, \overline{\pi}_2) = \Sigma^{-1}(\alpha - \mathbf{r}), \quad \overline{\theta} = \frac{1}{2}(\alpha - \mathbf{r})^{tr}\Sigma^{-1}(\alpha - \mathbf{r}).$$

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Since $z_i u_{z_i}$ is continuous from Theorem 1 (i), f is continuous.

In addition, as u is Lipschitz concave and satisfies the growth condition $\lim_{|z|\to\infty} u(z)/\ell(z) = 1$, we can apply the same argument as in the proof of Theorem 2.4 in Chen and Dai (2013) to infer that there are bounded functions $l_i^{\pm} : \mathbb{R} \mapsto \mathbb{R}$ and intervals $[b_i^{\pm}, s_i^{\pm}]$ for $i = 1, 2$ such that the characterizations in (2.12) and (2.13) hold. $\square$

5. Proof of Theorem 2

In this section, we prove Theorem 2. We begin with the derivation of the formal expansion.

For convenience, we recall the problem (2.15) for U

$$\begin{cases} -\sigma_1^2 U_{\xi_1\xi_1} - \sigma_2^2 U_{\xi_2\xi_2} = \Theta U + 2\rho\sigma_1\sigma_2 U_{\xi_1\xi_2} & \text{in } \ \mathbf{NT}_\xi, \\ U_{\xi_i} + (\beta_i + k_{ij} z_i) U = 0 & \text{on } \ \Gamma_{ij}, \\ U_{\xi_i\xi_i} + [k_{ij} z_i - (\beta_i + k_{ij} z_i)^2] U = 0 & \text{on } \ \Gamma_{ij} \end{cases}$$

where k_{ij} are defined in (2.16), and $\Theta := 2\theta - \sum_{ij} \sigma_{ij}\beta_i\beta_j$ is a constant being part of the unknown, $\mathbf{NT}_\xi$ and Γ_{ij} are the unknown no-transaction region and free boundary under new variables having the shape described in (2.12) and (2.13).

Here we remark that the first boundary condition in (2.15) is derived from $u_{z_i} = k_{ij}$ on Γ_{ij}. The second boundary condition in (2.15) is derived from $u_{z_i z_i} = 0$ and the following calculation on Γ_{ij}:

$$\begin{aligned} 0 = -z_i^2 u_{z_i z_i} &= -z_i (z_i u_{z_i})_{z_i} + z_i u_i = \frac{U_{\xi_i\xi_i}}{U} - \frac{U_{\xi_i}^2}{U^2} - \frac{U_{\xi_i}}{U} - \beta_i \\ &= \frac{U_{\xi_i\xi_i} - [\beta_i + k_{ij} z_i]^2 U + k_{ij} z_i U}{U}. \end{aligned}$$

Since $(z_i)_{\xi_i} = z_i$, the second boundary condition in (2.15) can also be written as

$$[U_{\xi_i} + (\beta_i + z_i k_{ij}) U]_{\xi_i} = 0 \quad \text{on } \ \Gamma_{ij}. \tag{5.1}$$

5.1. The Formal Expansion for Small ρ

Now we seek expansions in the form of (2.19)-(2.21). First, in $\mathbf{NT}_\xi$, we have

$$\begin{aligned} 0 &= -\sigma_1^2 U^0_{\xi_1\xi_1} - \sigma_2^2 U^0_{\xi_2\xi_2} - \Theta^0 U^0 \\ &\quad + \rho\Big\{ - \sigma_1^2 \hat{U}_{\xi_1\xi_1} - \sigma_2^2 \hat{U}_{\xi_2\xi_2} - \Theta^0 \hat{U} - \hat{\Theta} U^0 - 2\sigma_1\sigma_2 U^0_{\xi_1\xi_2} \Big\} + O(\rho^2). \end{aligned}$$

Second, the first set of boundary conditions gives the following:

$$\begin{aligned} 0 &= U_{\xi_i} + (\beta_i + k_{ij} z_i) U \Big|_{\xi_i = b_{ij} + \rho \hat{l}_{ij} + O(\rho^2)} \\ &= [U^0_{\xi_i} + (\beta_i^0 + k_{ij} z_i) U^0] \Big|_{\xi_i = b_{ij}} \\ &\quad + \rho\Big\{ \hat{l}_{ij} [U^0_{\xi_i} + (\beta_i^0 + k_{ij} z_i) U^0]_{\xi_i} + [\hat{U}_{\xi_i} + (\beta_i^0 + k_{ij} z_i)\zeta + \hat{\beta}_i U^0] \Big\} \Big|_{\xi_i = b_{ij}} + O(\rho^2) \end{aligned}$$

by using the boundary condition for U^0 and the identity (5.1).

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Third, the second set of boundary conditions can be written as

$$\begin{aligned} 0 &= [U_{\xi_i} + (\beta_i + k_{ij} z_i) U]_{\xi_i}\Big|_{\xi_i = b_{ij} + \rho \hat{l}_{ij} + o(\rho^2)} \\ &= [U^0_{\xi_i} + (\beta^0_i + k_{ij} z_i) U^0]_{\xi_i}\Big|_{\xi_i = b_{ij}} \\ &\quad + \rho \Big\{ \hat{l}_{ij} [U^0_{\xi_i} + (\beta^0_i + k_{ij} z_i) U^0]_{\xi_i \xi_i} + [\hat{U}_{\xi_i} + (\beta^0_i + k_{ij} z_i) \hat{U} + \hat{\beta}_i U^0]_{\xi_i} \Big\}\Big|_{\xi_i = b_{ij}} + O(\rho^2). \end{aligned}$$

Therefore, for the *zeroth*-order $\{\Theta^0, b_{ij}, U^0\}$, we have

$$\begin{cases} -\sigma_1^2 U^0_{\xi_1\xi_1} - \sigma_2^2 U^0_{\xi_2\xi_2} = \Theta^0 U^0 & \text{in } \mathbf{NT}^0_\xi = (b_{11}, b_{12}) \times (b_{21}, b_{22}), \\ U^0_{\xi_i} + (\beta^0_i + k_{ij} z_i) U^0 = 0 & \text{on } \Gamma^0_{ij} = \partial \mathbf{NT}^0_\xi \cap \{\xi_i = b_{ij}\}, \\ [U^0_{\xi_i} + (\beta^0_i + k_{ij} z_i) U^0]_{\xi_i} = 0 & \text{on } \Gamma^0_{ij}. \end{cases} \tag{5.2}$$

And, for the *leading* order $\{\hat{\Theta}, \hat{l}_{ij}, \hat{U}\}$, we have

$$\begin{cases} -\sigma_1^2 \hat{U}_{\xi_1\xi_1} - \sigma_2^2 \hat{U}_{\xi_2\xi_2} - \Theta^0 \hat{U} = \hat{\Theta} U^0 + 2\sigma_1\sigma_2 U^0_{\xi_1\xi_2} & \text{in } \mathbf{NT}^0_\xi, \\ \hat{U}_{\xi_i} + (\beta^0_i + k_{ij} z_i) \hat{U} + \hat{\beta}_i U^0 = 0 & \text{on } \Gamma^0_{ij}, \end{cases} \tag{5.3}$$

and

$$\hat{l}_{ij}(\breve{\xi}_i) = - \frac{[\hat{U}_{\xi_i} + (\beta^0_i + k_{ij} z_i) \hat{U} + \hat{\beta}_i U^0]_{\xi_i}}{[U^0_{\xi_i} + (\beta^0_i + k_{ij} z_i) U^0]_{\xi_i \xi_i}}\Big|_{\xi_i = b_{ij}} \quad \text{where } \breve{\xi}_1 := \xi_2,\ \breve{\xi}_2 := \xi_1. \tag{5.4}$$

5.2. **The Zeroth Order**

Note that the zeroth-order is equivalent to the case $\rho = 0$. The unique solution $U = U^0$ of (5.2) can be obtained by separation of variables:

$$U^0(\xi) = U_1(\xi_1) U_2(\xi_2), \qquad \Theta^0 = \sigma_1^2 \Theta_1 + \sigma_2^2 \Theta_2,$$

where $\{U_i, \Theta_i\}$, together with $\{b_{i1}, b_{i2}\}$, is the solution to the one-dimensional problem in terms of the new variables (ξ_1, ξ_2).[7]

To simplify exposition, in the rest of this subsection, let us denote by

$$\{U, \Theta, b_1, b_2, k_1, k_2, \beta, \xi\} = \{U_i, \Theta_i, b_{i1}, b_{i2}, k_{i1}, k_{i2}, \beta_i, \xi_i\} \quad \text{for either } i = 1 \text{ or } 2,$$

and

$$\gamma_j = \beta + k_j a_j \quad \text{for } j = 1, 2.$$

Then the one-dimensional problem is to find $(U, \Theta, b_1, b_2, a_1, a_2)$ such that $b_1 \le b_2$ and that

$$\begin{cases} -U'' = \Theta U & \text{in } (b_1, b_2), \\ U' + \gamma_j U = 0 & \text{at } b_j = \ln |a_j|, \quad j = 1, 2, \\ U'' + [k_j a_j - (\beta + k_j a_j)^2] U = 0 & \text{at } b_j, \qquad j = 1, 2, \end{cases} \tag{5.5}$$

This is equivalent to finding (U, Θ, b_1, b_2) such that

$$\begin{gathered} -U''(\xi) = \Theta U(\xi), \quad U(\xi) \neq 0 \qquad \forall\, \xi \in [b_1, b_2], \\ b_j = \ln \tfrac{|\gamma_j - \beta|}{k_j}, \quad U'(b_j) + \gamma_j U(b_j) = 0, \quad \gamma_j^2 - \gamma_j + \Theta + \beta = 0, \quad j = 1, 2. \end{gathered}$$

[7] In Appendix B, we present some basic properties for the above one-dimensional eigenvalue problem. We will use one of the properties in subsequent analysis. It is also worth pointing out that in the one-dimensional case, Guasoni and Muhle-Karbe (2015) conduct an asymptotic analysis through shadow price.

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This implies that a_j, b_j, γ_j are determined by $\Theta \leqslant 1/4 - \beta$. From (B.2) in Appendix B and the requirement $\ln|a_1| = b_1 \le b_2 = \ln|a_2|$, we must have that either $a_2 < a_1 < 0$ or $0 < a_1 < a_2$, leading to the choice

$$\gamma_1 = \frac{1 - \sqrt{1-4\beta-4\Theta}}{2}, \qquad \gamma_2 = \frac{1 + \sqrt{1-4\beta-4\Theta}}{2},$$

$$a_j = \frac{\gamma_j - \beta}{k_j} = \frac{1-2\beta+(-1)^j\sqrt{1-4\beta-4\Theta}}{2k_j} = \frac{m}{k_j} + (-1)^j\frac{\sqrt{1-4\beta-4\Theta}}{2k_j},$$

where $m = 1/2 - \beta$. Again, since either $a_2 < a_1 < 0$ or $0 < a_1 < a_2$, a straightforward calculation shows that $\Theta \geq -\beta^2$. So, this implies that $\Theta \in (-\beta^2, 1/4 - \beta)$.

Next, note that

$$b_j = \ln\frac{|\gamma_j - \beta|}{k_j} = \ln\Big|\frac{1-2\beta+(-1)^j\sqrt{1-4\beta-4\Theta}}{2k_j}\Big|.$$

This gives

$$b_2 - b_1 = A + \ln\Big|\frac{1-2\beta+\sqrt{1-4\beta-4\Theta}}{1-2\beta-\sqrt{1-4\beta-4\Theta}}\Big|, \tag{5.6}$$

where

$$A = \ln\frac{k_1}{k_2} = \ln\frac{1+\lambda}{1-\mu}.$$

Suppose $\Theta \neq 0$. Up to a constant multiple, the solution is given by

$$U(\xi) = \cos\Big(\sqrt{\Theta}[\xi - b_1] + \operatorname{arccot}\frac{\sqrt{\Theta}}{\gamma_1}\Big),$$

where the boundary condition $U'(b_2) + \gamma_2 U(b_2) = 0$ gives

$$b_2 - b_1 = \frac{1}{\sqrt{\Theta}}\Big(\operatorname{arccot}\frac{\sqrt{\Theta}}{\gamma_2} - \operatorname{arccot}\frac{\sqrt{\Theta}}{\gamma_1}\Big).$$

Here we use the convention that if $\Theta < 0$, then $\sqrt{\Theta} = \mathbf{i}\sqrt{-\Theta}$. Also, for $x \in \mathbb{R}$,

$$\cos(\mathbf{i}x) = \cosh(x), \quad \cot(\mathbf{i}x) = -\mathbf{i}\coth x, \qquad \operatorname{arccot}(-\mathbf{i}\coth x) = \mathbf{i}x.$$

In view of (5.6), for $A > 0$ and $\beta \neq 1/2$, we obtain a solution if and only if $\Theta \in (-\beta^2, 1/4-\beta)$ is a solution of

$$A = f(\beta, \Theta), \tag{5.7}$$

where

$$\begin{aligned} f(\beta,\Theta) \quad := \quad & \ln\frac{1-2\beta-\sqrt{1-4\beta-4\Theta}}{1-2\beta+\sqrt{1-4\beta-4\Theta}} \\ & + \frac{1}{\sqrt{\Theta}}\Big(\operatorname{arccot}\frac{2\sqrt{\Theta}}{1+\sqrt{1-4\beta-4\Theta}} - \operatorname{arccot}\frac{2\sqrt{\Theta}}{1-\sqrt{1-4\beta-4\Theta}}\Big). \end{aligned}$$

To sum up, we have the following lemma:

Lemma 5.1. *Assume that condition (2.11) holds. Then problem (5.2) has a unique solution $\{\Theta^0, b_{ij}, U^0\}$ having the forms of (2.22), (2.23), and (2.24) stated in Theorem 2.*

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In addition, based on this separability, the expression for $\hat{l}_{ij}$ in (5.4) can be simplified as follows. First of all, since $(z_i)_{\xi_i} = z_i$,

$$
\begin{aligned}
& [\hat{U}_{\xi_i} + (\beta_i^0 + k_{ij}z_i)\hat{U} + \hat{\beta}_i U^0]_{\xi_i}\Big|_{\xi_i=b_{ij}} \\
= \; & \hat{U}_{\xi_i\xi_i} + (\beta_i^0 + k_{ij}z_i)\hat{U}_{\xi_i} + k_{ij}z_i\hat{U} + \hat{\beta}_i U^0_{\xi_i}\Big|_{\xi_i=b_{ij}} \\
= \; & \hat{U}_{\xi_i\xi_i} + (-[\beta_i^0 + k_{ij}z_i]^2 + k_{ij}z_i\ )\hat{U} - 2(\beta_i^0 + k_{ij}z_i)\hat{\beta}_i U^0\Big|_{\xi_i=b_{ij}} \\
= \; & [\hat{U}_{\xi_i\xi_i} + \Theta_i\hat{U}] - 2\gamma_{ij}\hat{\beta}_i U_1U_2\Big|_{\xi_i=b_{ij}},
\end{aligned}
$$

where we have used the fact that $\gamma_{ij} = \beta_i^0 + k_{ij}z_i$ is the roots of $-\gamma^2 + \gamma - \beta_i^0 = \Theta_i$. Next, using $U^0 = U_1(\xi_1)U_2(\xi_2)$ and $(U_i^0)_{\xi_i\xi_i} = -\Theta_i U_i^0$ we have

$$
\begin{aligned}
& [U^0_{\xi_i} + (\beta_i^0 + k_{ij}z_i)U^0]_{\xi_i\xi_i}\Big|_{\xi_i=b_{ij}} \\
= \; & U^0_{\xi_i\xi_i\xi_i} + (\beta_i^0 + k_{ij}z_i)U^0_{\xi_i\xi_i} + k_{ij}z_i[2U^0_{\xi_i} + U^0]\Big|_{\xi_i=b_{ij}} \\
= \; & -\Theta_i[U^0_{\xi_i} + (\beta_i^0 + k_{ij}z_i)U^0] + k_{ij}z_i[1 - 2(\beta_i^0 + k_{ij}z_i)]U^0\Big|_{\xi_i=b_{ij}} \\
= \; & [\gamma_{ij} - \beta_i^0][1 - 2\gamma_{ij}]U_1U_2|_{\xi_i=b_{ij}}.
\end{aligned}
$$

Hence, the second set of boundary conditions can be written as

$$
\begin{aligned}
\hat{l}_{1j}(\xi_2) &= -\frac{\hat{U}_{\xi_1\xi_1}(b_{1j},\xi_2) + \Theta_1\hat{U}(b_{1j},\xi_2)}{H_{1j}\ U_2(\xi_2)} + \frac{2\gamma_{1j}\hat{\beta}_1}{(\gamma_{1j} - \beta_1^0)(1 - 2\gamma_{1j})} \quad \forall\, \xi_2 \in [b_{21}, b_{22}], \\
\hat{l}_{2j}(\xi_1) &= -\frac{\hat{U}_{\xi_2\xi_2}(\xi_1,b_{2j}) + \Theta_2\hat{U}(\xi_1,b_{2j})}{H_{2j}\ U_1(\xi_1)} + \frac{2\gamma_{2j}\hat{\beta}_2}{(\gamma_{2j} - \beta_2^0)(1 - 2\gamma_{2j})} \quad \forall\, \xi_1 \in [b_{11}, b_{12}],
\end{aligned}
$$

where H_{ij} are positive constants given by

$$
H_{ij} = (\gamma_{ij} - \beta_i^0)(1 - 2\gamma_{ij})U_i(b_{ij}), \quad \text{for } i, j = 1, 2.
$$

5.3. **The Leading Order**

In this subsection, we consider the problem (5.3) and (5.4) regarding the leading order terms $\hat{\Theta}$, $\hat{l}_{ij}$, and $\hat{U}$. We have the following result.

Lemma 5.2. *Assume condition* (2.11) *holds, then the problem* (5.3) *and* (5.4) *has a unique solution* $\{\hat{\Theta}, \hat{l}_{ij}, \hat{U}\}$ *having forms* (2.26), (2.27), (2.28), *and* (2.29).

Proof. Recall that the first variation $\hat{U}$ is the solution of the elliptic equation

$$
\begin{cases}
-\sigma_1^2\hat{U}_{\xi_1\xi_1} - \sigma_2^2\hat{U}_{\xi_2\xi_2} - \Theta^0\hat{U} = \hat{\Theta}U_1U_2 + 2\sigma_1\sigma_2U_1'U_2' & \text{in } \mathbf{NT}^0_\xi. \\
\hat{U}_{\xi_i} + \gamma_{ij}\hat{U} = -\hat{\beta}_i U^0 & \text{on } \Gamma^0_{ij},
\end{cases} \tag{5.8}
$$

where $\gamma_{ij} = \beta_i^0 + k_{ij}z_i$.

To solve the above linear partial differential equation, we introduce $\phi(\xi) = \hat{U}(\xi) + \sum_{i=1}^2 \hat{\beta}_i\xi_i$. Then ϕ solves the following equation

$$
\begin{cases}
\mathbf{L}\phi = \hat{\Theta}U_1U_2 + 2\sigma_1\sigma_2U_1'U_2' - 2\sigma_1^2\hat{\beta}_1U_1'U_2 - 2\sigma_2^2\hat{\beta}_2U_1U_2' & \text{in } \mathbf{NT}^0_\xi. \\
\phi_{\xi_i} + \gamma_{ij}\phi = 0 & \text{on } \Gamma^0_{ij},
\end{cases} \tag{5.9}
$$

where $\mathbf{L}$ is a self-adjoint elliptic operator

$$
\mathbf{L} := -\sigma_1^2\partial_{\xi_1\xi_1} - \sigma_2^2\partial_{\xi_2\xi_2} - \Theta^0\mathbf{I}.
$$

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Note that $U^0 := U_1U_2$ is a solution of the homogeneous equation, i.e., U^0 is the principal eigenfunction with a zero principal eigenvalue of the corresponding self-adjoint elliptic operator $\mathbf{L}$ associated with the mixed boundary conditions. The solvability condition gives the constant

$$\hat{\Theta} = \frac{2\int_{\mathbf{NT}^0_\xi} U_1U_2[\sigma_1^2\hat{\beta}_1U_1'U_2 + \sigma_2^2\hat{\beta}_2U_1U_2' - \sigma_1\sigma_2U_1'U_2']d\xi}{\int_{\mathbf{NT}^0_\xi} U_1^2U_2^2 d\xi}.$$

Since the solution of the original problem is unique up to a constant multiple, we can fix the multiple by requiring $\phi \perp U^0$, i.e.,

$$c_{00} := \frac{\int_{\mathbf{NT}^0_\xi} \phi U_1U_2 d\xi}{\sqrt{\int_{\mathbf{NT}^0_\xi} U_1^2U_2^2 d\xi}} = 0.$$

The solution of (5.9) can be obtained by Fourier series. For this, denote by $\{\Theta_{ip}, \psi_{ip}\}_{p=0}^\infty$ for $i = 1, 2$ all the eigenpairs of the eigenvalue problem (2.31) in Remark 2.2, i.e.,

$$\begin{cases} -\psi_{ip}'' - \Theta_i\psi_{ip} = \Theta_{ip}\psi_{ip} & \text{in } [b_{i1}, b_{i2}], \\ \psi_{ip}'(b_{ij}) + \gamma_{ij}\psi_{ip}(b_{ij}) = 0, & j = 1, 2. \\ \int_{b_{i1}}^{b_{i2}} \psi_{ip}^2(x)dx = 1. \end{cases}$$

In fact, $\{\Theta_{ip}, \psi_{ip}\}_{p=0}^\infty$ has an explicit form, where Θ_{ip} is the root of the following algebraic equation

$$\sqrt{\Theta_i + \Theta_{ip}}(b_{i2} - b_{i1}) = p\pi + \operatorname{arccot}\frac{\sqrt{\Theta_i + \Theta_{ip}}}{\gamma_{i2}} - \operatorname{arccot}\frac{\sqrt{\Theta_i + \Theta_{ip}}}{\gamma_{i1}}, \tag{5.10}$$

and ψ_{ip} is given explicitly by

$$\psi_{ip}(\xi_i) = \cos\left(\sqrt{\Theta_i + \Theta_{ip}}(\xi_i - b_{i1}) + \operatorname{arccot}\frac{\sqrt{\Theta_i + \Theta_{ip}}}{\gamma_{i1}}\right).$$

Now, having $\{\Theta_{ip}, \psi_{ip}\}_{p=0}^\infty$ at hand, solutions of the eigenvalue problem

$$\mathbf{L}\tilde{\psi} = \tilde{\theta}\tilde{\psi} \quad \text{with } \tilde{\psi}_{\xi_i} + \gamma_{ij}\tilde{\psi} = 0,$$

are

$$\tilde{\theta}_{pq} := \sigma_1^2\Theta_{1p} + \sigma_2^2\Theta_{2q}, \qquad \tilde{\psi}_{pq} := \psi_{1p}\psi_{2q}, \quad \text{for } p, q = 0, 1, \cdots.$$

Note that $\psi_{i0} = U_i/\|U_i\|_{L^2}$ and that $\Theta_{i0} = 0$. The solution of (5.9) thus can be written as

$$\phi(\xi_1, \xi_2) = \sum_{p=0}^\infty\sum_{q=0}^\infty c_{pq}\psi_{1p}(\xi_1)\psi_{2q}(\xi_2),$$

and the following calculation holds:

$$\phi_{\xi_i\xi_i}(\xi_1, \xi_2) + \Theta_i\phi(\xi_1, \xi_2) = -\sum_{p=0}^\infty\sum_{q=0}^\infty c_{pq}\Theta_{ip}\psi_{1p}(\xi_1)\psi_{2q}(\xi_2) \quad \text{for } i = 1, 2.$$

Here the coefficients are given by $c_{00} = 0$ and (by using $\psi_{1p}\psi_{2q} \perp U_1U_2$)

$$c_{pq} = \frac{2\sigma_1\sigma_2\int_{b_{11}}^{b_{12}} U_1'(\xi_1)\psi_{1p}(\xi_1)d\xi_1 \int_{b_{21}}^{b_{22}} U_2'(\xi_2)\psi_{2q}(\xi_2)d\xi_2}{\sigma_1^2\Theta_{1p} + \sigma_2^2\Theta_{2q}} \quad \text{for } (p, q) \neq (0, 0). \tag{5.11}$$

Consequently, we can obtain the solution of (5.8) by setting

$$\hat{U}(\xi) = \phi(\xi) - \sum_{i=1}^2 \hat{\beta}_i\xi_i.$$

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A further calculation shows that

$$\begin{aligned}\hat{l}_{1j}(\xi_2) &= -\frac{\hat{U}_{\xi_1\xi_1}(b_{1j},\xi_2)+\Theta_1\hat{U}(b_{1j},\xi_2)}{H_{1j}\,U_2(\xi_2)}+\frac{2\gamma_{1j}\hat{\beta}_1}{(\gamma_{1j}-\beta_1^0)(1-2\gamma_{1j})}\\ &= \sum_{q=0}^{\infty}\frac{\psi_{2q}(\xi_2)}{U_2(\xi_2)}\sum_{p=1}^{\infty}c_{pq}\Theta_{1p}\frac{\psi_{1p}(b_{1j})}{H_{1j}}\end{aligned}$$

for $j=1,2$ and $\xi_2\in[b_{21},b_{22}]$. Similarly,

$$\begin{aligned}\hat{l}_{2j}(\xi_1) &= -\frac{\hat{U}_{\xi_2\xi_2}(\xi_1,b_{2j})+\Theta_2\hat{U}(\xi_1,b_{2j})}{H_{2j}\,U_1(\xi_1)}+\frac{2\gamma_{2j}\hat{\beta}_2}{(\gamma_{2j}-\beta_2^0)(1-2\gamma_{2j})}\\ &= \sum_{p=0}^{\infty}\frac{\psi_{1p}(\xi_1)}{U_1(\xi_1)}\sum_{q=1}^{\infty}c_{pq}\Theta_{2q}\frac{\psi_{2q}(b_{2j})}{H_{2j}}\end{aligned}$$

for $j=1,2$ and $\xi_1\in[b_{11},b_{12}]$. Note that at the corner point,

$$\begin{aligned}\hat{l}'_{1j}(\xi_2)|_{\xi_2=b_{2j}} &= \sum_{q=0}^{\infty}\frac{\psi'_{2q}U_2-\psi_{2q}U'_2}{U_2^2}\Big|_{\xi_2=b_{2j}}\sum_{p=1}^{\infty}c_{pq}\theta_{1p}\frac{\psi_{1p}(b_{1j})}{H_{1j}}\\ &= \sum_{q=0}^{\infty}\frac{[\psi'_{2q}+\gamma_{2j}\psi_{2q}]U_2-\psi_{2q}[U'_2+\gamma_{2j}U_2]}{U_2^2}\Big|_{\xi_2=b_{2j}}\sum_{p=1}^{\infty}c_{pq}\Theta_{1p}\frac{\psi_{1p}(b_{1j})}{H_{1j}}=0.\end{aligned}$$

Similarly, $\hat{l}'_{2j}(b_{1j})=0$. This completes the proof of the lemma. $\square$

Proof of Theorem 2. Theorem 2 follows immediately from Lemmas 5.1 and 5.2. $\square$

Appendix A. A Heuristic Derivation of (2.4)

In this appendix, we provide a heuristic derivation of the eigenvalue problem (2.4), which corresponds to the ergodic control (2.2).

Following Guasoni and Muhle-Karbe (2015), we begin with the following finite horizon problem

$$V(x,y,t;T)=\sup_{(L,M)\in\mathcal{C}_t(x,y)}\mathbb{E}_t^{x,y}\Big[-\exp\Big(-\nu[X_T+\ell(Y_T)]\Big)\Big] \tag{A.1}$$

subject to dynamics (2.1) with the initial value that $(X_{t^-},Y_{t^-})=(x,y)\in\mathbb{R}\times\mathbb{R}^2$. Here $\mathcal{C}_t(x,y)$ denotes all admissible strategies starting from the position (x,y) at time t, and $\mathbb{E}_t^{x,y}$ is an expectation operator conditional on $(X_{t^-},Y_{t^-})=(x,y)$.

Then by dynamic programming principle, V solves the following HJB equation

$$\min_{i\in\{1,2\}}\min\{-V_t-\mathscr{L}V,\ (1+\lambda_i)V_x-V_{y_i},\ -(1-\mu_i)V_x+V_{y_i}\}=0\quad\text{in }\mathbb{R}^3\times[0,T) \tag{A.2}$$

where the linear operator $\mathscr{L}$ is defined by

$$\mathscr{L}V=\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_{ij}y_iy_jV_{y_iy_j}+\sum_i\alpha_iy_iV_{y_i}+rxV_x.$$

Motivated by the classic solution in Merton (1969), we introduce the following transformation

$$V(x,y,t;T)=-e^{-\nu e^{r(T-t)}x-\theta(T-t)-u(z,t;T)}, \tag{A.3}$$

where $z=\nu e^{r(T-t)}y$ is the wealth invested in the risky assets adjusted by both risk-free rate and risk aversion, and θ is chosen such that $\lim_{T-t\to\infty}u_t(z,t;T)=0$. A straightforward calculation leads to the HJB equation (2.4) by sending $T-t$ to infinity.

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To establish a rigorous linkage between the eigenvalue problem (2.4) and the ergodic control problem (2.2), one needs to prove either a verification theorem or a dynamic programming principle. Guasoni and Muhle-Karbe (2015) prove the verification theorem for the single risky asset case (or the case of multiple uncorrelated risky assets) for which an analytical solution is available. For the correlated risky assets case, due to the absence of an analytical solution, the proof of the verification theorem requires a certain regularity of the solution to (2.4). We leave it for future study.

Appendix B. Basic Properties in the One-Dimensional Case of Problem (2.4)

The one-dimensional infinite horizon problem of (2.4) is to find an interval $[a_1, a_2] \subset \mathbb{R}$, an (eigenvalue) $\tilde{\theta} \in \mathbb{R}$, and a function $u \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ such that

$$\begin{cases} \tilde{\theta} - z^2 u'' + (zu')^2 - 2mzu' = 0 & \forall z \in [a_1, a_2], \\ k_2 \leqslant u'(z) \leqslant k_1 & \forall\, z \in (a_1, a_2), \\ u'(z) - k_1 = 0 \leqslant \tilde{\theta} + (zk_1)^2 - 2mk_1 z, & \forall\, z < a_1, \\ u'(z) - k_2 = 0 \leqslant \tilde{\theta} + (zk_2)^2 - 2mk_2 z & \forall\, z > a_2 \end{cases} \tag{B.1}$$

where

$$m = \frac{\alpha - r}{\sigma^2}, \qquad k_1 = 1 + \lambda \in [1, \infty), \qquad k_2 = 1 - \mu \in (0, 1].$$

The original eigenvalue θ relates $\tilde{\theta}$ by $\theta = \frac{\sigma^2}{2}\tilde{\theta}$.

Notice first the following basic facts:

(1) If u is a solution, then for any constant C, $u + C$ is also a solution. Hence, we normalize it by assuming

$$u(0) = 0.$$

(2) There are two trivial cases:

(i) $\lambda = 0 = \mu$: then up to an additive constant, the solution is given by

$$u(z) = z \quad \forall\, z \in \mathbb{R}, \qquad a_1 = a_2 = m, \qquad \tilde{\theta} = m^2.$$

(ii) $m = 0$: Then up to an additive constant, the solution is given by

$$a_1 = 0,\ a_2 = 0,\ \tilde{\theta} = 0, \quad u(z) = \begin{cases} (1-\mu)z & \text{if } z \geqslant 0, \\ (1+\lambda)z & \text{if } z < 0. \end{cases}$$

(3) If $0 \in [a_1, a_2]$, then we must have $\tilde{\theta} = 0$ or equivalently $\theta = 0$. From $zu'' = zu'^2 - 2mu'$ in $(a_1, a_2) \setminus \{0\}$ and $k_2 \leqslant u' \leqslant k_1$ we conclude first that $a_1 = a_2 = 0$ and then $m = 0$. Hence, we must have $m \neq 0$ and $0 \notin [a_1, a_2]$. Moreover, since $\theta \geq 0$, u is concave, and $u' \geq k_2 > 0$, the equation $\tilde{\theta} - z^2 u'' + (zu')^2 - 2mzu' = 0$ implies that $mz > 0$ for $z \in [a_1, a_2]$. Thus, we have

$$m > 0 \Longrightarrow a_2 > a_1 > 0, \quad \text{and} \quad m < 0 \Longrightarrow a_1 < a_2 < 0. \tag{B.2}$$

(4) The differential equation at a_i and the differential inequality imply that $a_i^2 u''(a_i) \geqslant 0$. On the other hand, $u'(a_1) = k_1$ is a global maximum and $u'(a_2) = k_2$ is a global minimum of u', so $a_i^2 u''(a_i) = 0$.

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