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Min DAI, Yaoting LEI and Hong LIU

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NUS Risk Management Institute
21 HENG MUI KENG TERRACE, #04-03 I3 BUILDING, SINGAPORE 119613

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Min Dai † Yaoting Lei ‡ Hong Liu §

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† National University of Singapore. Tel: (65) 6516 2754. Email: matdm@nus.edu.sg. Dai acknowledges support from Singapore AcRF grants (Grant No. R-703-000-032-112, R-146-000-306-114, and R-146-000-311-114) and the National Natural Science Foundation of China (Grant No. 11671292).

‡ Nanchang University, 999 Xuefu Road, Nanchang, China 330031, Email: leiyaoting@ncu.edu.cn. Lei acknowledges support from the National Natural Science Foundation of China (Grant No. 71701086).

§ Washington University in St. Louis, One Brookings Drive, St. Louis, MO 63130. Tel: (314) 935 5883. Email: liuh@wustl.edu.

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Although prevalent in the financial markets, transaction costs have been largely ignored in the existing literature on the impact of capital gains tax. We develop a dynamic portfolio model that incorporates both transaction costs and annually payable capital gains taxes. We theoretically characterize and numerically compute the optimal tax-timing strategy. We show that our model can help explain the puzzle that even when investors face equal long-term/short-term capital gains tax rates, they may still defer realizing large capital losses and vary the realization strategies of capital gains and losses across months in a calendar year. In addition, our model provides some unique, empirically testable implications, such as (1) investors with capital losses may purchase a large number of shares without realizing any losses; and (2) investors' trading may *increase* with transaction costs.

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1 Introduction

Transaction costs are prevalent in financial markets, and it is well known that even tiny transaction cost can have a large impact on optimal trading strategies. However, most of the extant literature on dynamic portfolio choice with capital gains taxes ignores transaction costs. In addition, apart from the fact that capital gains taxes are paid at the end of each calendar year (the "year-end tax rule"), the extant literature assumes that capital gains taxes are paid immediately after a sale. As a result of these simplifications, the literature shows that it is optimal to realize capital losses immediately, and that the tax-timing strategy is time-invariant if long- and short-term tax rates are the same (e.g., Constantinides, 1986; Dai, Liu, Yang, and Zhong, 2015). However, empirical evidence suggests the opposite: even investors who are facing equal long- and short-term tax rates often even defer large capital losses[1] and tend to realize losses towards the end of a year (e.g., Slemrod, 1982).

To help explain the above empirical evidence, we propose a tax-timing model that incorporates transaction costs and the year-end tax rule. More specifically, we develop a continuous-time optimal investment and consumption model for a constant relative risk aversion (CRRA) investor who faces symmetric capital gains taxes (i.e., equal long- and short-term tax rates), proportional transaction costs, and the year-end tax rule. According to U.S. tax law, the maximum tax rebate that an investor can claim per year is capped at $\$3,000$, and the remainder can be carried forward indefinitely. This motivates us to consider two polar-opposite cases: the full rebate (FR) case in which there is no limitation for tax rebate for losses; and the full carry-over (FC) case in which capital losses can only be carried

[1] In most countries, the capital gains tax rates for a long-term investment (i.e., a holding period greater than one year) and a short-term investment (i.e., a holding period less than one year) are the same (e.g., Japan, UK, France, Canada, etc.). For the U.S., these rates were also the same for lower-income investors from 1988–1997. Many investors who face equal long-term/short-term tax rates may even defer large losses. For example, in 1997, investors with an annual income below $20,000 realized over $1.39 billion losses, but only after they became long term (see for example, *https://www.irs.gov/statistics/soi-tax-stats-sales-of-capital-assets-reported-on-individual-tax-returns*); and in the U.K., investors realized over 10 million pounds of losses that were longer than 6 months in 2002–2003 (see for example, *https://www.gov.uk/government/collections/capital-gains-tax-statistics*).

forward to offset future capital gains. These two cases offer reasonable bounds for the limited use of capital losses for a tax rebate. The FR case applies better to investors with low stock investment (e.g., low-wealth investors), while the FC case applies better to investors with high stock investment (e.g., high-wealth investors).

We show that the optimal tax-timing policy derived from our model is largely consistent with the above empirical evidence. For example, our model reveals that even with tiny transaction costs (e.g., 0.5%), investors who face equal long- and short-term tax rates may defer large capital losses, especially when the interest rate is low. This is because realization of losses incurs transaction costs that may outweigh the time-value of tax rebate for losses. This result can help explain why investors may hold stocks with large capital losses without realizing those losses for a long period of time. Moreover, because of the year-end rule, investors tend to realize capital gains more often and in larger amounts in early months, and they tend to realize more capital losses in late months in the FR case. This is because at the beginning of a year, the present value of a tax payment or tax rebate, which will occur at the end of the year, is smaller. On the other hand, in contrast to discrete time models with annual trading frequency, where investors, by definition, can only realize gains and losses at the end or the beginning of a year, our continuous-time model implies that it is optimal for investors to realize gains and losses across time within the year when risk exposure is sufficiently far from the ideal level or losses are sufficiently high.

In the existing models with capital gains tax, when an investor has capital losses, it is never optimal to buy a lump-sum amount of the stock without first realizing some losses. In contrast, we show that this trading strategy may be optimal in the presence of transaction costs. Intuitively, realizing losses incurs additional transaction costs. As a result, when losses are not sufficiently large but risk exposure is too low, it is optimal for the investor to buy a lump-sum amount of the stock to bring the risk exposure closer to the ideal risk-exposure level. Selling some first would incur additional transaction costs. Furthermore, surprisingly, it may also be optimal to buy a lump-sum amount of the stock when transaction costs are high but to not trade when transaction costs are small; thus, trading may *increase* with transaction costs.

The rationale for this counterintuitive result is as follows. An investor trades off risk exposure, tax rebates, and transaction costs. With lower transaction costs, it is optimal for an investor to realize all the losses (i.e., a wash sale) to receive the tax rebate even when those losses are relatively small. Therefore, when the losses are smaller than but close to the threshold for a wash sale, an investor may choose not to trade so that he can save transaction costs for imminent wash-sale trades, even when the exposure is far below the ideal level. In contrast, it is not optimal for an investor with higher transaction costs to realize small losses; as a result, the risk exposure may be far below the ideal level *and* the losses are much lower than the threshold for a realization of losses. Therefore, it may be optimal for an investor facing higher transaction costs and the same magnitude of losses to purchase a lump-sum amount of the stock to bring her risk exposure closer to the ideal level.

It is well-known that in the presence of proportional transaction costs or capital gains taxes, there is a no-trading region wherein investors do not trade, and transactions only occur at the boundaries of this no-trading region. The resulting optimal policy for realizing gains is to sell a minimal amount to remain inside the no-trading region. In contrast, we find that a lump-sum sale to realize gains may be optimal in our model. The optimality of a lump-sum gain realization is due to the year-end rule, because the optimal trading boundaries are one-year periodic, and the sell boundary jumps down from December 31st to January 1st due to the jump of the present value of the tax payment for gains realized between the two dates.

Our model thus provides some unique, empirically testable implications for investors facing symmetric long- and short-term tax rates. For example, (1) they may make a lump-sum purchase without realizing any of the current losses; and (2) they may choose to trade when transaction costs are large, but not to trade when they are small.

Literature review. Merton (1969, 1971) pioneered the study of continuous-time portfolio choice under a frictionless market. Merton's strategy involves continuous portfolio rebalancing, which can be costly and sub-optimal in the presence of market frictions. Magill and Constantinides (1976) introduced transaction costs to Merton's model and show the existence of a no-trading region. From then on, portfolio choice with transaction costs has been extensively studied; see, for example, Constantinides (1986), Davis and Norman (1990), Shreve and Soner (1994), Liu and Loewenstein (2002), Liu (2004), Dai and Yi (2009), Dai, Jin, and Liu (2011), and so forth.

Capital gains taxes are imposed in many countries, and tax-timing is an important part of investment strategy. Due to the strong path dependency incurred by the "exact tax basis,"[2] most of the existing literature on portfolio choice with capital gains tax has been restricted to a discrete-time framework with a limited number of time steps.[3] The continuous-time or truly multiple-period portfolio choice models with capital gains tax were not studied until the work of Dammon, Spatt, and Zhang (2001), who propose a multiple-period binomial portfolio choice model with the approximation of the exact tax basis using the "average tax basis" to overcome the curse of dimensionality.[4] Ben Tahar, Soner, and Touzi (2007, 2010) further formulated a continuous-time version of the model proposed by Dammon, Spatt, and Zhang (2001). Cai, Chen, and Dai (2018) extended this to non-constant investment opportunities and employed an asymptotic analysis to investigate the effect of capital gains tax on optimal portfolio allocation. The above literature shows that investors may defer the realization of capital gains, but that they should realize capital losses immediately; moreover,

[2]A capital gain or loss for a particular share sold is computed by the difference between the sale price and the original purchase price of this share. Therefore, one needs to keep track of the exact original purchase price of each share.

[3]See e.g., Constantinides (1983, 1984), Dybvig and Koo (1996), and DeMiguel and Uppal (2005).

[4]The average tax basis is the weighted average of past purchase prices of the shares held and is actually used in Canada. It is a reasonably good approximation of the exact tax basis (cf. DeMiguel and Uppal, 2005; and Dai, Liu, Yang, and Zhong, 2015).

the optimal tax-timing strategy with an infinite horizon is time-invariant. These findings were based on three assumptions: transaction costs are absent, tax rebates are unlimited, and capital gains taxes are paid immediately after sale (i.e., the "instant tax rule"). In contrast, empirical studies indicated that investors may defer realizing even large capital losses (Wilson and Liddell, 2010) and more investors realize capital losses in December (Slemrod, 1982; Constantinides, 1984).

Dai, Liu, Yang, and Zhong (2015) develop an optimal tax-timing model in which the long-term capital gains tax rates are lower than the short-term tax rates. Their model helps to explain why many investors not only defer short-term capital losses to the long term, but also defer large long-term capital gains and losses. However, in contrast to our model, Dai, Liu, Yang, and Zhong (2015) could not explain why investors who face the same long- and short-term tax rates may also defer large capital losses.

The remainder of the paper is organized as follows: Section 2 is devoted to the model setup. In Section 3, we provide some theoretical results. We conduct an extensive numerical analysis on optimal strategies in Section 4 and provide some unique empirically testable predictions. We conclude in Section 5. All proofs are presented in the Appendix.

2 The Model

We consider a financial market consisting of only two investment assets. The first asset is a risk-free money market account growing at a constant, continuously compounded interest rate of r. The second asset is a risky stock (or index) paying a constant, continuous dividend yield of δ. The ex-dividend stock price process S_t evolves according to a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where $\mu + \delta > r$ and $\sigma > 0$ are constants representing, respectively, the expected rate of return and the volatility of the stock, and B_t is a standard one-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$.

The investor can buy the stock at the ask price of $(1+\theta)S_t$ and sell it at the bid price of $(1-\alpha)S_t$, where $\theta \in [0, \infty)$ and $\alpha \in [0, 1)$ are constants representing proportional transaction cost rates for purchasing and selling the stock, respectively.

The sales of the stock are subject to a capital gains tax at a constant rate of $\tau \in [0, 1)$. As in most of the existing literature (e.g., Dammon and Spatt, 1996; Marekwica, 2012; Dai, Liu, Yang, and Zhong, 2015; Ehling, Gallmeyer, Srivastava, Tompaidis, and Yang, 2018), in calculating capital gains taxes, we use the average purchase cost as the cost basis. Different from most of the continuous-time models with capital gains tax but consistent with real tax law, we assume capital gains taxes are paid at the end of each calendar year. We term this case as the year-end tax (YT) case. For comparison, we term the case where capital gains taxes are paid immediately after sale, as assumed in the existing literature, as the instant tax (IT) case. We consider both the FR case, wherein an investor can use all capital losses to offset taxable ordinary income, and the FC case, wherein the investor can only carry forward capital losses to offset future gains and/or income. The FR case applies better to lower-income investors whose capital losses are likely less than \$3,000 per year, while the FC case is more suitable for higher-income investors for whom capital losses can be much more than \$3,000 per year and a tax rebate (which is capped at $\$3,000 \times 39.6\% = \$1,188$ per year) is relatively unimportant.

We assume that interest and dividend are also subject to taxes at constant tax rates $\tau_i \in [0, 1)$ and $\tau_d \in [0, 1)$, respectively. These taxes are paid at the end of each calendar year in the YT case and immediately in the IT case.

We denote by x_t the after-tax dollar amount invested in the money market account, y_t the dollar amount invested in the stock, and k_t the cost basis for the stock holding. Then we have the following dynamics for x_t, y_t and k_t:[5]

$$dx_t = \left[\tilde{r}(t;\lambda)x_t + \tilde{\delta}(t;\lambda)y_t - c_t\right]dt - (1+\theta)dL_t + f(t,0,y_{t-},k_{t-};l,\lambda)dM_t, \tag{1}$$

$$dy_t = \mu y_{t-}dt + \sigma y_{t-}dB_t + dL_t - y_{t-}dM_t, \tag{2}$$

$$dk_t = (1+\theta)dL_t - k_{t-}dM_t + l(k_{t-} - (1-\alpha)y_{t-})^+ dM_t, \tag{3}$$

where

$$f(t,x,y,k;l,\lambda) = x + (1-\alpha)y - g(t;\lambda)\tau\left[(1-l)((1-\alpha)y-k) + l((1-\alpha)y-k)^+\right]$$

is the total wealth after liquidation, c_t is the consumption rate,

$$\tilde{r}(t;\lambda) = r(1-\tau_i g(t;\lambda)), \quad \tilde{\delta}(t;\lambda) = \delta(1-\tau_d g(t;\lambda)),$$

$$g(t;\lambda) = \begin{cases} 1, & \text{for } \lambda = 0, \\ \frac{1}{\tau_i + (1-\tau_i)e^{r(\lceil t\rceil - t)}}, & \text{for } \lambda = 1, \end{cases}$$

$\lambda = 0$ or 1 corresponds to the IT case or the YT case respectively, $l = 0$ or 1 corresponds to the FR case or the FC case respectively, $\lceil t\rceil$ represents the smallest integer not less than t, and L_t and M_t are two right-continuous, non-negative and non-decreasing $\mathscr{F}_t-$adapted processes with $L_{0-} = M_{0-} = 0$, denoting respectively the dollar amount purchased and the proportion of the stock position sold.

For the case $\lambda = 1$, $g(t;1)$ represents discounting factor whose expression comes from the fact that one dollar in the bank account at time t will become

$$e^{r(\lceil t\rceil - t)} - \tau_i(e^{r(\lceil t\rceil - t)} - 1) = \tau_i + (1-\tau_i)e^{r(\lceil t\rceil - t)} = \frac{1}{g(t;1)}$$

[5]See Appendix A.1 for the derivation of the dynamics of x_t in the YT case.

at the end of the year after tax deductions.

The investor's problem is to choose an admissible strategy to maximize the expected utility from intertemporal consumption. The value function at time t can be defined as

$$V(t,x,y,k) = \max_{\mathscr{A}_t(x,y,k)} \mathbb{E}_t^{x,y,k} \left[\int_t^{\infty} e^{-\beta(s-t)} U(c_s) ds \right], \tag{4}$$

where $\beta > 0$ is a constant discount factor,

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0,\, \gamma \neq 1,$$

is a CRRA utility function with a constant risk aversion factor γ, and $\mathscr{A}_t(x,y,k)$ is the set of all admissible strategies (c, L, M) such that the unique solution of (1)-(3) with $(x_t, y_t, k_t) = (x, y, k)$ satisfies

$$f(s, x_s, y_s, k_s; l, \lambda) \geq 0,\ \forall s \geq t.$$

We always assume $\beta > (1-\gamma)\left[r + \frac{(\mu+\delta-r)^2}{2\gamma\sigma^2}\right]$, which ensures that the value function given in (4) is well-defined (see, e.g., Ben Tahar, Soner, and Touzi (2010)).

3 Theoretical Analysis

In this section, we conduct a theoretical analysis that facilitates our subsequent analysis.

The value function for the IT case (i.e., $\lambda = 0$) is clearly time-independent. The following proposition indicates that if $\lambda = 1$, the value function is one-year periodic, which allows us to construct an efficient numerical algorithm.

***Proposition* 1** (Time homogeneity and periodicity)**:** If $\lambda = 0$, the value function V is time-independent, i.e., $V(t,x,y,k) \equiv V(x,y,k)$. If $\lambda = 1$, the value function V is one-year periodic, that is,

$$V(t,x,y,k) = V(t+1,x,y,k), \quad \forall\, t \geq 0,\, (x,y,k) \in \bar{\mathscr{S}_t},$$

where $\mathscr{S}_t = \{(x,y,k) | y > 0, k > 0, f(t,x,y,k;l,\lambda) > 0\}$.

Since closed-form solutions are usually unavailable in the presence of transaction costs and/or capital gains tax, we will seek a numerical solution. This motivates us to consider the following Hamilton-Jacobi-Bellman (HJB) equation that the value function satisfies:

$$\max\Big\{V_t + \mathcal{L}_0 V,\, \mathcal{B}_0 V,\, \mathcal{S}_0 V\Big\} = 0, \quad t \geq 0,\, (x,y,k) \in \mathscr{S}_t, \tag{5}$$

where

$$\begin{aligned}
\mathcal{L}_0 V &= \frac{1}{2}\sigma^2 y^2 V_{yy} + \mu y V_y + [\tilde{r}(t;\lambda)x + \tilde{\delta}(t;\lambda)y]V_x - \beta V + U^*(V_x), && (6)\\
\mathcal{B}_0 V &= -(1+\theta)V_x + V_y + (1+\theta)V_k, && (7)\\
\mathcal{S}_0 V &= f(t,0,y,k;l,\lambda)V_x - yV_y - \big[k - l(k-(1-\alpha)y)^+\big]V_k, && (8)
\end{aligned}$$

and

$$U^*(q) = \sup_{c>0}\big\{U(c) - cq\big\} = \frac{\gamma}{1-\gamma} q^{1-1/\gamma}.$$

Using the homogeneity property of the utility function, we can reduce the dimensionality of the problem by the following transformation:

$$V(t,x,y,k) = y^{1-\gamma}\Phi(t,z,b), \quad z = \frac{x}{y}, \quad b = \frac{k}{y}. \tag{9}$$

Thus, it can be verified that $\Phi(t,z,b)$ satisfies

$$\max\Big\{\Phi_t + \mathcal{L}\Phi,\, \mathcal{B}\Phi,\, \mathcal{S}\Phi\Big\} = 0, \quad t \geq 0,\, (z,b) \in \mathscr{T}_t, \tag{10}$$

where $\mathscr{T}_t = \{(z, b) | f(t, z, 1, b; l, \lambda) > 0, b > 0\}$, and

$$
\begin{aligned}
\mathcal{L}\Phi &= \frac{1}{2}\sigma^2 z^2 \Phi_{zz} + \frac{1}{2}\sigma^2 b^2 \Phi_{bb} + \sigma^2 z b \Phi_{zb} - \left[\left(\mu - \tilde{r}(t;\lambda) - \gamma\sigma^2\right) z - \tilde{\delta}(t;\lambda)\right] \Phi_z \\
&\quad -(\mu - \gamma\sigma^2) b \Phi_b + \left[(1-\gamma)\left(\mu - \frac{1}{2}\gamma\sigma^2\right) - \beta\right] \Phi + U^*(\Phi_z), \\
\mathcal{B}\Phi &= (1-\gamma)\Phi - (1+\theta+z)\Phi_z + (1+\theta-b)\Phi_b, \\
\mathcal{S}\Phi &= -(1-\gamma)\Phi + f(t, z, 1, b; l, \lambda)\Phi_z + l(b-1+\alpha)^+ \Phi_b.
\end{aligned}
$$

For each time $t \geq 0$, the domain $\mathscr{T}_t$ can be partitioned into three regions: the sell region (SR), the buy region (BR), and the no-trading region (NTR), which are defined as follows:

$$
\begin{aligned}
\mathrm{SR}_t &= \left\{(z, b) \in \mathscr{T}_t \middle| \mathcal{S}\Phi(t, z, b) = 0\right\}, \\
\mathrm{BR}_t &= \left\{(z, b) \in \mathscr{T}_t \middle| \mathcal{B}\Phi(t, z, b) = 0\right\}, \\
\mathrm{NTR}_t &= \left\{(z, b) \in \mathscr{T}_t \middle| \mathcal{S}\Phi(t, z, b) < 0 \text{ and } \mathcal{B}\Phi(t, z, b) < 0\right\}.
\end{aligned}
$$

By Proposition 1, we infer that if $\lambda = 0$, then Φ is time-independent and its numerical algorithm is similar to that in Dai, Liu, Yang, and Zhong (2015); if $\lambda = 1$, then Φ is one-year periodic:

$$
\Phi(t+1, z, b) = \Phi(t, z, b),
$$

which motivates us to use the following algorithm to solve equation (10) in the case of $\lambda = 1$.

Algorithm of finding the numerical solution for $\lambda = 1$

In the case of $\lambda = 1$, we use the following iterative algorithm to solve for Φ in one period $t \in [0, 1]$:

1. Give an initial guess of $\Phi_0(0, z, b)$, and set $j = 1$.

2. At the j-th iteration, solve for $\Phi_j(t,z,b)$ using (10), $t \in [0,1)$, with terminal condition

$$\Phi_j(1,z,b) = \Phi_{j-1}(0,z,b).$$

3. If $|\Phi_j(1,z,b) - \Phi_j(0,z,b)| <$ tolerance, then stop and set $\Phi = \Phi_j$; otherwise, set $j = j+1$, and go to Step 2.

In recent years, many countries have adopted low-interest-rate policies to stimulate their economies. We next present a result for the FR case with a zero interest rate.

Proposition 2: Consider the FR case with a zero interest rate: i.e., $l = 0$ and $r = 0$.

1. In the absence of transaction costs i.e., $\theta = \alpha = 0$, the value function has a closed form

$$V(x,y,k) = \frac{K^{-\gamma}}{1-\gamma}(x + (1-\tau)y + \tau k)^{1-\gamma},$$

where

$$K = \frac{\beta}{\gamma} - \frac{1-\gamma}{\gamma}\left[\frac{((1-\tau)\mu + (1-\tau_d)\delta)^2}{2\gamma(1-\tau)^2\sigma^2}\right].$$

The optimal strategy is to invest and consume a constant fraction of after-tax liquidation wealth: i.e., $\frac{y}{x+y-\tau(y-k)} = \frac{(1-\tau)\mu+(1-\tau_d)\delta}{\gamma(1-\tau)^2\sigma^2}$ and $\frac{c}{x+y-\tau(y-k)} = K$.

2. In the presence of transaction costs, the investor's problem reduces to the following tax-free portfolio-choice problem with transaction costs: to maximize $\mathbb{E}_t^{x,y,k}\left[\int_t^{\infty} e^{-\beta(s-t)}U(c_s)ds\right]$ by choosing admissible strategy $(\hat{c}, \hat{L}, \hat{M})$ subject to $\hat{x}_t + (1-\alpha)\hat{y}_t \geq 0$ and $\hat{c} \geq 0$ for all $t \geq 0$, where

$$d\hat{x}_t = \left(\frac{1-\tau_d}{1-\tau}\delta\hat{y}_t - c_t\right)dt - (1+\theta)d\hat{L}_t + (1-\alpha)\hat{y}_{t-}d\hat{M}_t, \quad (11)$$

$$d\hat{y}_t = \mu\hat{y}_{t-}dt + \sigma\hat{y}_{t-}dB_t + d\hat{L}_t - \hat{y}_{t-}d\hat{M}_t. \quad (12)$$

Moreover, if $(\hat{c}^*, \hat{L}^*, \hat{M}^*)$ is the optimal strategy of the tax-free problem, then the strategy $(c^*, L^*, M^*) = (\hat{c}^*, \hat{L}^*/(1-\tau), \hat{M}^*)$ is optimal for the original taxable problem.

Proposition 2 shows that for the FR case with zero interest rate, the investor's optimization problem can be transformed into a problem without capital gains tax after adjusting certain parameter values. In particular, in the absence of transaction costs with a zero interest rate, there is no benefit to deferring capital gains realization, so it is optimal to realize gains and losses continuously. It is worth noting that for the FC case, the optimization problem cannot be transformed into one without capital gains tax, even when the interest rate is zero, because gains and losses are treated asymmetrically.

With a positive interest rate, we will elaborate on the optimal trading strategies characterized by BR, SR, and NTR in the next section.

4 Numerical Analysis

In this section, we will conduct extensive numerical analysis of the investor's optimal trading strategy (i.e., the tax-timing strategy). Unless otherwise stated, we will always use the default parameter values summarized in Table 1, most of which are the same as those in Dai, Liu, Yang, and Zhong (2015), except $\tau = 0.15$ and $\theta = \alpha = 0.005$.

4.1 How do transaction costs affect the tax-timing strategy?

In this subsection, we will restrict attention to the IT rule. Let us first consider the FR case.

4.1.1 The FR and IT case

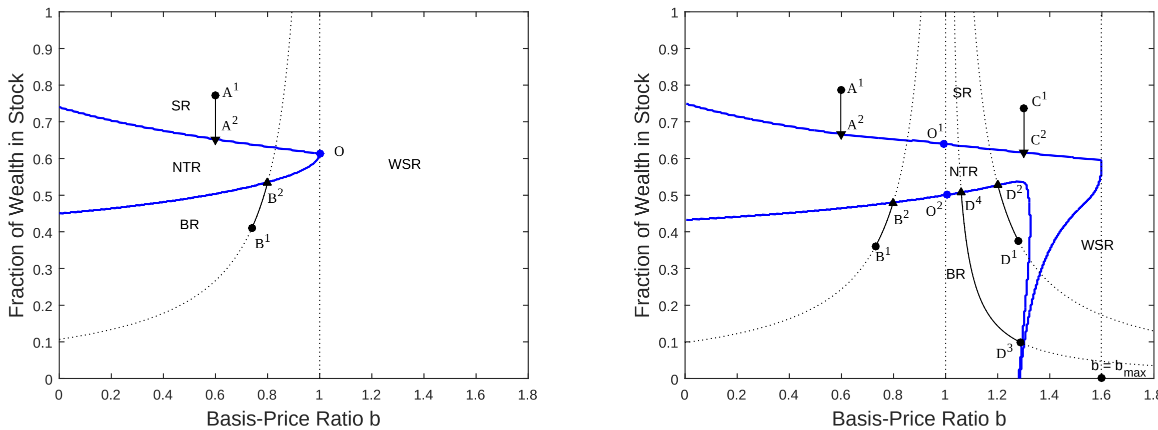
We begin our analysis by briefly recalling the case with zero transaction costs.[6] Figure 1(a) plots the optimal trading strategy in the $b-\pi$ plane, where b (the horizon axis) is the basis-

[6]This case was studied in Dai, Liu, Yang, and Zhong (2015) and Cai, Chen, and Dai (2018).

Table 1: The Default Values of the Parameters

Variable	Symbol	Default Value
Interest rate	r	0.03
Dividend yield	δ	0.02
Expected stock return	μ	0.07
Stock return volatility	σ	0.2
Subjective discount rate	β	0.01
Risk aversion factor	γ	3
Tax rate for interest	τ_i	0.35
Tax rate for dividend	τ_d	0.35
Tax rate for capital gains on sales of stock	τ	0.15
Proportional transaction cost for purchase	θ	0.005
Proportional transaction cost for sale	α	0.005

price ratio, and $\pi = \frac{y}{f(t,x,y,k;l,\lambda)}$ (the vertical axis) denotes the fraction of the after-tax total wealth in the stock. It can be seen that when there is a capital gain (i.e., $b < 1$), there are three regions: SR, NTR, and BR, which are split by two trading boundaries. The optimal sell boundary is between SR and NTR, and the optimal buy boundary is between NTR and BR. The transaction direction in BR (or SR), which is marked in the figure, is determined by the characteristic line of the first-order equation $\mathcal{B}\Phi = 0$ (or $\mathcal{S}\Phi = 0$). Note that the characteristic line in SR is vertical to the b-axis, which indicates that selling stock does not change the basis-price ratio b. Thus, in SR, where (b, π) is above the sell boundary, the investor will sell vertically downward to reach the sell boundary (e.g., from point A^1 to A^2). The characteristic line in BR is a curve, as purchasing stock upon gains will increase the basis-price ratio b. As a consequence, in BR, where (b, π) is below the buy boundary, the investor will buy to reach the buy boundary along an upper-right direction (e.g., from B^1 to B^2). If (b, π) is in NTR, no transaction occurs; NTR indicates that the investor may defer realizing capital gains to save the time value of capital gains taxes. It is worthwhile to point out that the buy and sell boundaries intersect with $b = 1$ at point O, and the whole region for $b > 1$ is the "wash sale" region (WSR), within which the investor sells all the stock and repurchases to reach the position O. This implies that whenever there are capital losses, the investor immediately realizes all the losses, and hence, in the absence of transaction cost, the investor will not defer realizing capital losses.



(a) $\theta = 0$ (b) $\theta = 0.005$

Figure 1: Optimal trading boundaries (the FR and IT case)

Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

We will now investigate the impact of transaction costs on the investor's trading strategy. Figure 1(b) shows the optimal trading boundaries with transaction cost rate $\theta = \alpha = 0.5\%$. When there are capital gains (i.e., $b < 1$), there are still two boundaries, known as the optimal sell boundary and the optimal buy boundary, which split the region into SR, NTR, and BR. The sell (buy) boundary is higher (lower) than the corresponding boundary without transaction costs,[7] which implies that transaction costs cause the investor to defer the realization of capital gains even more. The sell boundary intercepts the line $b = 1 - \alpha$ at O^1, which implies that if the initial wealth is all in stock, then one should rebalance the portfolio to O^1. Similarly, the buy boundary interacts with the line $b = 1 + \theta$ at O^2, which indicates that if the initial wealth is all in a bank account, then one should rebalance the

[7] This can be more clearly seen in Figure 3.

portfolio to O^2. It is worthwhile to point out that point O^2 is also the optimal portfolio position to which the investor should rebalance after a wash sale.

Transaction costs qualitatively change the optimal trading strategy when there are capital losses. It can be seen that the NTR exists for $b > 1$, which suggests that transaction costs may lead the investor to defer the realization of capital losses. When the basis-price ratio is greater than a threshold value b_{max}, a wash sale is always optimal. However, when the basis-price ratio does not exceed b_{max}, the trading strategy is determined by a trade-off among maintaining reasonable risk exposure, saving transaction costs, and the desire to receive tax rebates early: if the fraction of wealth in stock π is sufficiently high, such that (b, π) is in the SR with capital losses, the investor should sell to the sell boundary (e.g., from C^1 to C^2), as liquidating such a large stock holding is too costly in the presence of transaction costs; if π is lower and the basis-price ratio is sufficiently large, a wash sale is still optimal, as transaction costs incurred by liquidating such a small stock holding can be compensated by the available tax rebates; and if both π and the basis-price ratio are sufficiently low, such that the current state (b, π) lies in the BR with capital losses, it is optimal for the investor to buy to reach the buy boundary along an upper-left direction (e.g., from D^1 to D^2), as determined by the characteristic line of the first order equation $\mathcal{B}\Phi = 0$ in this region.

The buy region on the $b > 1$ side illustrates a unique empirically testable prediction of our model: When the capital losses are not too great and the risk exposure is low, investors with capital losses may buy a large number of additional shares without selling some beforehand (e.g., from D^3 to D^4 in Figure 1(b)). This prediction is in sharp contrast with the prediction of all the existing models, wherein it is always optimal to sell first in order to realize some losses, and then purchase back some shares. The main driving force behind this new result is the transaction costs. In the presence of transaction costs, selling first then buying back incurs additional transaction costs, and these additional costs may outweigh earning the time-value of the tax rebate from a sale. As a result, to save the transaction costs, the investor may choose to buy additional shares without selling first to get the tax rebate. This result of an investor possibly buying a lump sum at a buy boundary is also the first in the literature on portfolio choice with proportional transaction costs. The existing literature has shown that it is optimal to make minimum transactions to remain in the no-transaction region. In the presence of transaction costs, a lump sum purchase at the buy boundary is optimal because the risk exposure at the boundary is too far from the optimal level due to deferring the realization of losses to save transaction costs.

Figure 2 plots the optimal trading boundaries with a zero interest rate. As shown by Proposition 2, in the absence of transaction costs, the investor should continuously realize capital gains or losses and maintain a constant fraction of wealth in stock as represented by the black dot at $b = 1$ in the figure; in the presence of transaction costs, the investor should keep her risk-exposure (i.e., the fraction of wealth in stock) within a certain range that is independent of the price-basis ratio, as represented by the solid line (i.e., buy boundary) and the dashed line (i.e., sell boundary) in the figure. As suggested by Proposition 2, these trading strategies are the same as the case without a capital gains tax after adjusting certain parameter values by the tax rate.

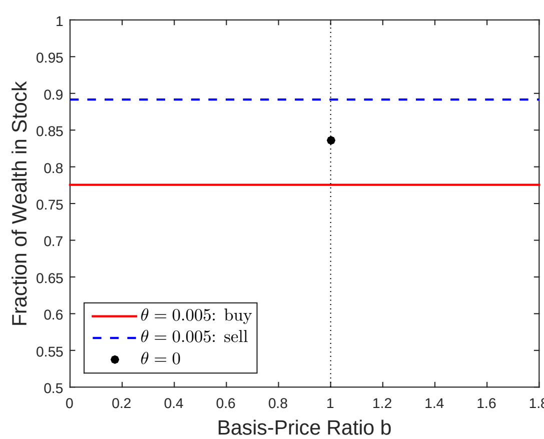


Figure 2: Optimal trading boundaries with zero interest rate (the FR case)
Parameter values: $r = 0$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

Now, we will investigate how the magnitude of transaction costs affects the investor's trading policy. In Figure 3, we observe that the NTR tends to expand as the transaction cost rate increases. However, the NTR with a 0.1% transaction cost rate is not completely contained in the NTR with a 0.5% transaction cost rate, which implies that there exists a scenario in which an investor with a larger transaction cost rate (0.5%) will trade the stock, while the one with a smaller transaction cost rate (0.1%) will not transact. This seems to be counter-intuitive. Indeed, an investor trades off risk exposure, tax rebates, and transaction costs. With lower transaction costs, it is optimal for an investor to realize all the losses to get the tax rebate, even when the losses are relatively small. Therefore, an investor with a lower transaction cost may choose not to trade so that she can save transaction costs for imminent wash-sale trades. In contrast, in Region A, an investor with a higher transaction cost will buy the minimum amount to reach the boundary because wash-sale trades are in the distant future, transacting a minimum amount does not incur large transaction costs, and it is important to maintain sufficient risk exposure.

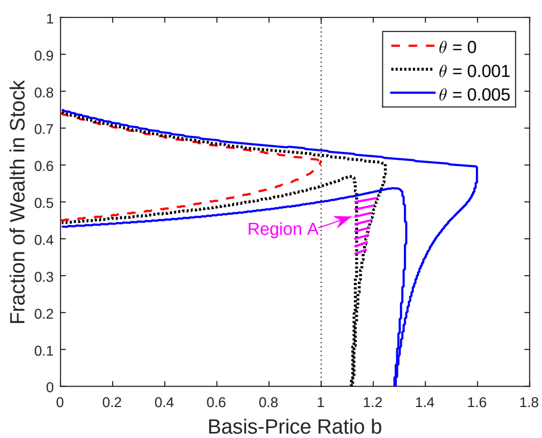


Figure 3: Optimal trading boundaries for different transaction costs (the FR and IT case) Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

The extent of the loss realization deferral can be measured by $b_{\max}$. Figure 4 plots the threshold value $b_{\max}$ against the transaction cost rate for different levels of tax rates in the FR case. Figure 4 shows that $b_{\max}$ is as large as 1.6 (i.e., a 60% loss) for a transaction cost rate of $\theta = \alpha = 0.5\%$ and a tax rate of $\tau = 0.15$, which indicates that investors may even defer realizing large capital losses with tiny transaction costs (i.e., 0.5%). We can also observe that $b_{\max}$ significantly increases with transaction cost rates and decreases with tax rates. This is because with a larger transaction cost rate or a smaller tax rate, investors need a higher tax rebate to offset the transaction costs incurred by a wash sale.

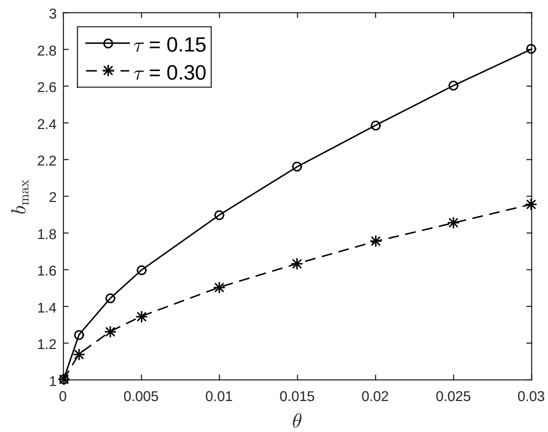


Figure 4: $b_{\max}$ against transaction cost rate (the FR and IT case)
Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, and $\alpha = \theta$.

To investigate how long the investor will defer capital losses in the presence of transaction costs, we use Monte-Carlo simulations to compute the average time to hold capital losses until holding gains or realizing losses. In Figure 5, the dashed line and the solid line represent the average duration of holding losses against transaction costs for a zero interest rate and a non-zero interest rate, respectively. It is not surprising that the higher the transaction costs, the longer the time of holding losses. In particular, even with a small transaction cost rate $\theta = \alpha = 0.5\%$ and a tax rate of $\tau = 0.15$, the average duration of holding losses would be two months. When the interest rate is lower, it is optimal to defer longer on average because the time value of a tax rebate is less with a lower interest rate.

We also use Monte-Carlo simulations to compute the average time to hold capital gains until holding losses or gains realization. In Figure 6, the dashed line and the solid line represent the average duration of holding gains against the transaction costs for a zero

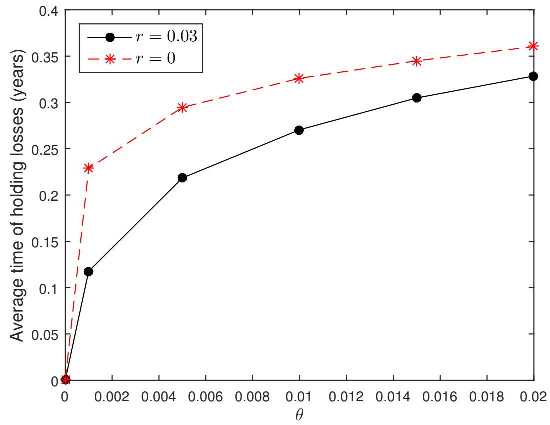


Figure 5: The average duration of holding losses (the IT and FR case)
Parameter values: $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

interest rate and a non-zero interest rate, respectively. It can be observed that the investor significantly defers capital gains in the presence of transaction costs. When the interest rate is lower, it is optimal to defer for a shorter period of time on average, because the time-value of the tax payment is less with a lower interest rate.

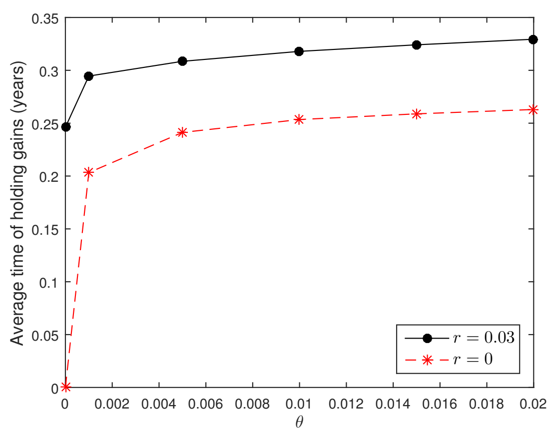
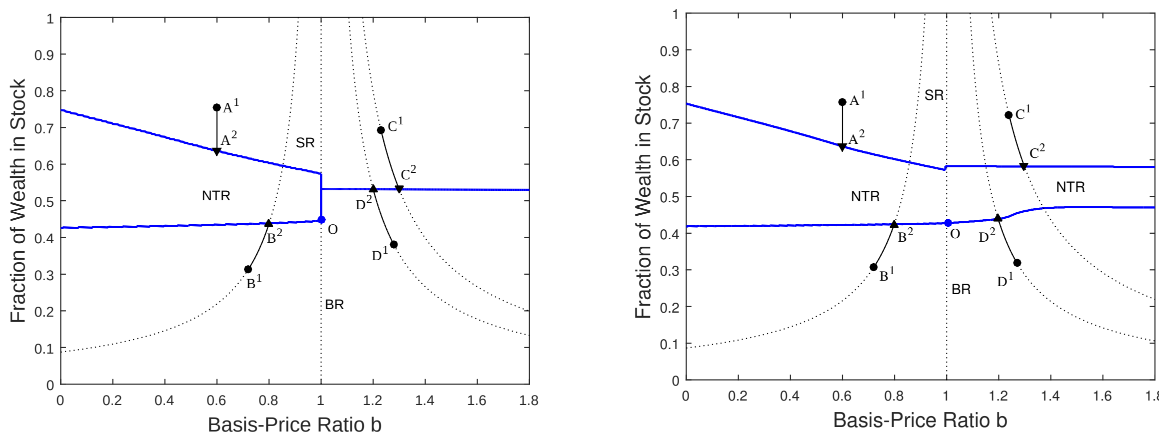


Figure 6: The average duration of holding gains (the IT and FR case)
Parameter values: $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

4.1.2 The FC and IT case

We now examine the FC case with the IT rule. Again, let us first recall the zero-transaction-costs case, which was also studied in Dai, Liu, Yang, and Zhong (2015). The optimal trading boundaries are shown in Figure 7(a): when there is a capital gain (i.e., $b < 1$), the optimal trading policy is similar to that in the FR case, except that the buy and sell boundaries intersect with $b = 1$ at two points; and when there is a capital loss (i.e., $b > 1$), there is an optimal fraction of wealth (represented by the blue line) at which level the investor will continuously trade to maintain (e.g., sell from point C^1 to C^2 or buy from D^1 to D^2 along the characteristic line). Bearing in mind that capital losses can only be carried forward to offset future capital gains, the blue trading line represents the optimal risk exposure that the investor would like to maintain against the basis-price ratio in the presence of capital losses. The existence of the trading line for $b > 1$ implies that the investor continuously realizes capital losses in the absence of transaction costs.



(a) $\theta = 0$ (b) $\theta = 0.005$

Figure 7: Optimal trading boundaries (the FC and IT case)
Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

In the presence of transaction costs, the optimal trading boundaries for the FC and IT case are presented in Figure 7(b). It can be observed that the NTR exists for all $b \geq 1$, which indicates that due to lacking the incentive of a tax rebate, the investor might defer realizing significant capital losses as long as her risk exposure does not exceed a reasonable range.

Figure 8 plots the optimal trading boundaries with a zero interest rate for the FC case with the IT rule; the solid and dashed lines represent the optimal trading boundaries with and without transaction costs, respectively. As shown in Proposition 2, with a zero interest rate and zero transaction costs, it is optimal to continuously realize gains and losses in the FR case. In contrast, Figure 8 shows that in the FC case, it is optimal to defer capital gains, even with a zero interest rate and zero transaction costs. This is because by deferring capital gains payment in some sample paths, the investor may never pay taxes, because in these paths, the subsequent losses can be used to offset the current gains; and when there are capital losses and no transaction costs, the investor continuously trade along the dashed line inside $b > 1$; and when there are transaction costs, the investor can defer both gains and losses. Overall, unlike in the FR case, a zero interest rate does not qualitatively change the optimal trading strategy in the FC case.

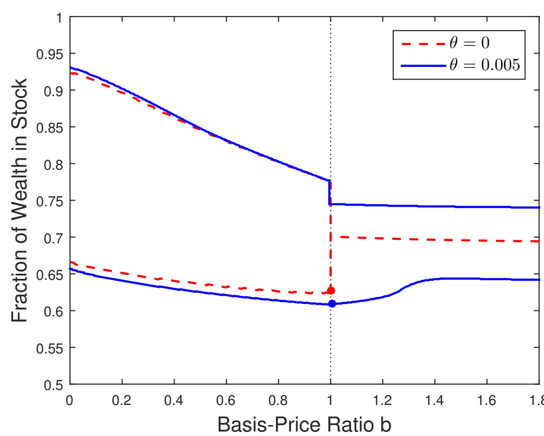


Figure 8: Optimal trading boundaries with zero interest rate (the FC and IT case)
Parameter values: $r = 0$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

As in the FR case, we use the Monte-Carlo simulation to compute the average duration from holding capital losses to subsequent holding gains or realizing (partial) losses. In Figure 9, the dashed and solid lines represent the average duration of holding losses against transaction costs for a zero interest rate and a non-zero interest rate, respectively. In the absence of transaction costs, the average duration must be zero, as the investor must continuously realize losses. As in the FR case, the average duration monotonically increases with the transaction cost rate. In particular, the duration can be four months long (0.3 year) with a small transaction cost rate of $\alpha = \theta = 0.5\%$. In contrast to the FR case, as the interest rate increases, the average holding period also increases. This is because, without rebate, it is important to keep greater losses to offset subsequent gains so that the investor can achieve better risk exposure without paying any tax.

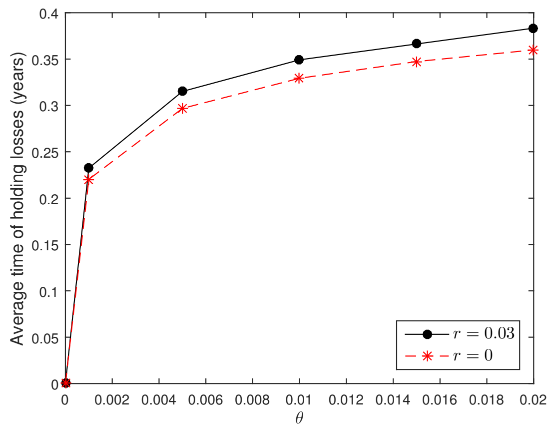


Figure 9: The average duration of holding losses (the IT and FC case)
Parameter values: $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

We also compute the duration of holding gains, as shown in Figure 10. This is qualitatively similar to the FR case.

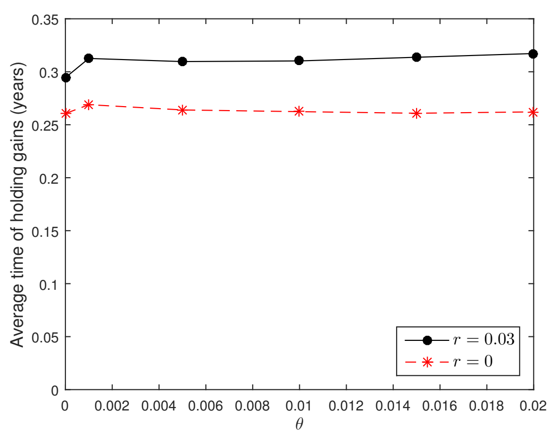


Figure 10: The average duration of holding gains (the IT and FC case)
Parameter values: $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

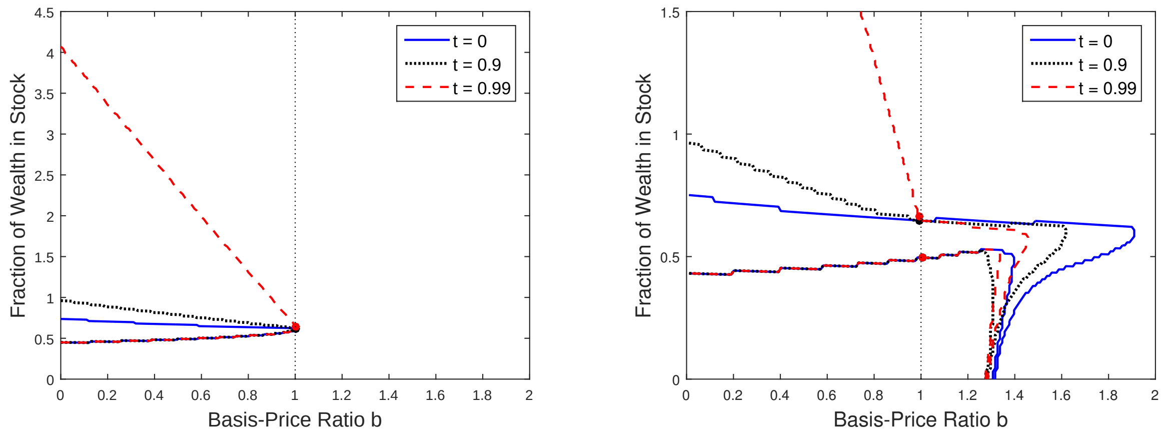
4.2 How does the year-end tax rule affect the tax-timing strategy?

Next, we will examine how the year-end tax rule affects the tax-timing strategy. In the following analysis, the year-end tax rule is assumed unless otherwise stated.

4.2.1 The FR and YT case

Let us first look at the FR case without transaction costs. Figure 11(a) plots the optimal trading boundaries without transaction costs at different times. As expected, the investor still realizes losses immediately, but the trading strategy in the presence of capital gains is no longer time-invariant. At a given time t, the trading boundaries have a pattern that is similar to those with the IT rule, and the investor does not defer any capital losses. Interestingly, the buy boundary seems insensitive to time, but the sell boundary dramatically rises as time goes by, especially near the end of the calendar year ($t = 1$), which suggests that the investor is more willing to realize capital gains in the early months of each calendar year, because the time-value of the tax payment to be paid at the end of the year is lower. In particular, the sell boundary at the beginning of the calendar year is significantly lower than at the end of a calendar year, which implies that the investor has a significant opportunity to realize a lump-sum capital gain at the beginning of the next calendar year (i.e., in January).

We now move to the FR case with transaction costs and plot the corresponding optimal trading boundaries in Figure 11(b). To see the boundaries more clearly, we change the range of the y-axis to $[0, 1.5]$. It can be observed that when there is a capital gain, the trading strategy is similar to when there are no transaction costs, which suggests that the investor is still inclined to realize capital gains in the early months of each calendar year and to realize a lump-sum gain in January. When there is a capital loss, as in the IT case, there also exists an NTR, which suggests that the investor may defer realization of capital losses. More importantly, the NTR shrinks in the presence of capital losses as time goes by, which implies that consistent with empirical studies, the investor has an incentive to realize capital losses



(a) $\theta = 0$ (b) $\theta = 0.005$

Figure 11: Optimal trading boundaries (the FR and YT case)
Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

in the late months (i.e., in December).

The tendency to realize gains at the beginning of a year and to realize losses at the end of a year is qualitatively similar to the prediction of a model with lower long-term capital gains tax rates than short-term tax rates (see for example, Dai, Liu, Yang, and Zhong, 2015). Therefore, our model predicts that even when the long-term tax rates are the same as the short-term tax rates, an investor tends to realize more gains at the beginning of the year and realize losses at the end of the year.

To illustrate the frequency and magnitude of capital gain and loss realizations across time with the YT rule, we simulate 100,000 sample paths for 1,440 trading dates with an initial wealth of $1 and report simulation results for the FR case in Tables 2 and 3.

Table 2 reports the average numbers of gain realizations and of states with gains, the ratio between the two, the percentage of gains realized, and the dollar amounts of the realized gain and tax payment for each month. It can be observed that while there is no significant difference among the numbers of the gain state in different months, the investor realizes more gains in the early months of a calendar year. In addition, the trading frequency, dollar amounts of realized gains, and tax payment all monotonically decrease from January to December. Moreover, the realized gains and tax payments in January are significantly higher than in other months, as a lump-sum gain realization can be optimal in January. On the other hand, with the IT rule, these measures are constant across time.

Table 3 reports the corresponding results for capital losses. In contrast to the gains realization, the loss realization difference across different months is small. Still, consistent with Figure 11(b), the investor realizes a higher percentage of the losses in December than in January.

Interestingly, as shown in Tables 2 and 3, we find that the monthly average amounts of realized gains and tax payments/rebates with the YT rule are similar to those obtained with the IT rule. This phenomenon is consistent with our finding in Appendix A.4, which shows that when implementing the YT rule, the certainty equivalent-wealth-loss incurred by the optimal strategy with the IT rule is marginal, despite the fact that the optimal strategies with two tax rules differ significantly.

4.2.2 The FC and YT case

We now turn to the FC case with the YT rule. We plot the optimal trading boundaries in Figure 12(a) for the no-transaction-cost case and in Figure 12(b) for the positive-transaction-cost case. Note that the range of the y-axis in Figure 12(b) is changed to $[0, 1.5]$, which is different from that in Figure 12(a), because we would like to show the boundaries more clearly in the loss region. It can be seen that when there are capital gains, the trading policies with or without transaction costs are similar to the corresponding trading policies in the FR case. In particular, investors are inclined to realize capital gains in the early months of each calendar year. When there are capital losses, however, the optimal policies with or without transaction costs are almost insensitive to time, which indicates that the investor is indifferent to which month to realize capital losses, as the investor no longer qualifies for tax rebates and all realized losses are equally carried forward, irrespective of the time of

Table 2: Simulation results: Gains realizations (the FR case)

This table reports the results of some statistics of capital gains realizations obtained by the Monte-Carlo simulation. Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta = 0.005$.

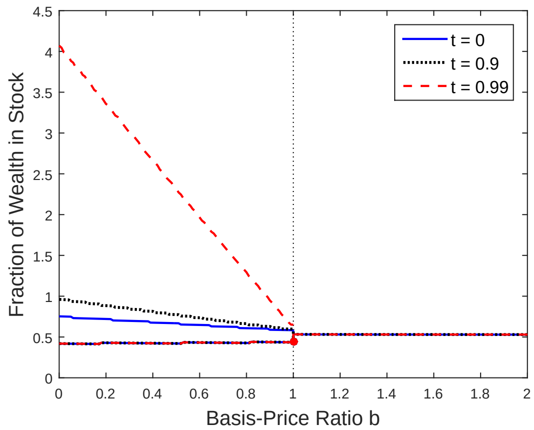
	No. of gain sell	No. of gain state	% of gain sell	% of gain realized	Amount of realized gain	Amount of tax payment
			Year-End Tax Rule			
Jan	18.29	105.93	17.26%	1.71%	0.4312	0.0488
Feb	10.93	105.71	10.34%	0.75%	0.1121	0.0123
Mar	9.53	105.90	9.00%	0.74%	0.0983	0.0109
Apr	8.73	106.07	8.23%	0.74%	0.0908	0.0101
May	8.18	106.21	7.70%	0.74%	0.0862	0.0097
Jun	7.48	106.37	7.04%	0.73%	0.0808	0.0092
Jul	7.21	106.50	6.77%	0.73%	0.0784	0.0090
Aug	6.46	106.65	6.05%	0.72%	0.0651	0.0073
Sep	4.83	106.81	4.52%	0.70%	0.0418	0.0043
Oct	2.03	106.96	1.90%	0.66%	0.0148	0.0014
Nov	0.23	107.13	0.21%	0.60%	0.0009	0.0000
Dec	0.01	107.33	0.01%	0.55%	0.0000	0.0000
Average	6.99	106.46	6.57%	0.78%	0.0917	0.0103
			Instant Tax Rule			
Monthly	8.93	106.51	8.38%	0.73%	0.0921	0.0104

Table 3: Simulation results: Losses realizations (the FR case)

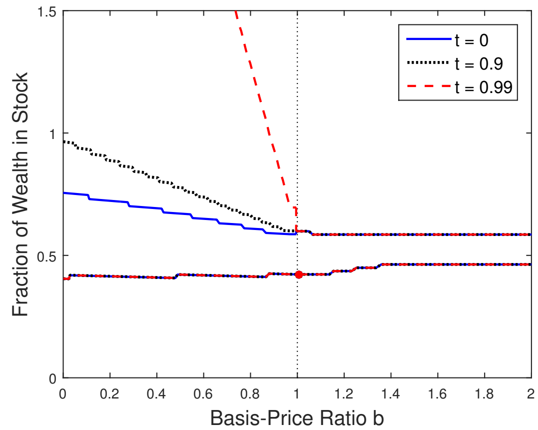
This table reports the results of some statistics of capital losses realizations obtained by the Monte-Carlo simulation. Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta = 0.005$.

	No. of loss sell	No. of wash sale	No. of loss state	% of loss sell	% of loss realized	Amount of realized loss	Amount of tax rebate
			Year-End Tax Rule				
Jan	0.0013	0.0006	14.07	0.01%	35%	0.0002	0.00002
Feb	0.0058	0.0051	14.29	0.04%	76%	0.0016	0.00016
Mar	0.0115	0.0105	14.10	0.08%	87%	0.0035	0.00033
Apr	0.0151	0.0142	13.93	0.11%	90%	0.0045	0.00044
May	0.0201	0.0191	13.79	0.15%	92%	0.0061	0.00059
Jun	0.0222	0.0213	13.63	0.16%	93%	0.0068	0.00066
Jul	0.0286	0.0277	13.50	0.21%	95%	0.0090	0.00085
Aug	0.0307	0.0298	13.35	0.23%	96%	0.0097	0.00090
Sep	0.0343	0.0333	13.19	0.26%	96%	0.0109	0.00099
Oct	0.0442	0.0432	13.04	0.34%	97%	0.0144	0.00127
Nov	0.0559	0.0550	12.87	0.43%	98%	0.0185	0.00157
Dec	0.1293	0.1271	12.67	1.02%	98%	0.0446	0.00335
Average	0.0333	0.0322	13.54	0.25%	88%	0.0108	0.00093
			Instant Tax Rule				
Monthly	0.0346	0.0335	13.41	0.26%	95%	0.0112	0.00096

realization. In addition, in contrast to the FR case, the investor can defer an unlimited amount of losses, as long as the risk exposure is close to the optimal exposure.



(a) $\theta = 0$



(b) $\theta = 0.005$

Figure 12: Optimal trading boundaries (the FC and YT case)
Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, $\tau = 0.15$, and $\alpha = \theta$.

5 Conclusion

In this paper, we propose an optimal consumption and investment model with both transaction costs and capital gains taxes, where we consider two tax rules: the IT rule and the YT rule. We show that our model can help explain the puzzling empirical evidence that investors who are facing equal long-term/short-term capital gains tax rates may also defer realizing large (and possibly unlimited) capital losses, and on average, investors realize capital losses more often in later months. In addition, our model provides unique empirically testable implications, such as (1) investors with capital losses may purchase a large number of shares without first realizing some losses; and (2) investors may choose to trade when transaction costs are high but to not trade when they are low.

Appendix

A.1 Derivation of dynamics (1) for $\lambda = 1$

Assume $\lambda = 1$. Before the year-end settlement, the investor records realized capital gains and losses; at the end of a calendar year, the investor pays year-end taxes (i.e., taxes on accumulative net realized capital gains). This is equivalent to discounting the amount of year-end taxes as cash-flows at times immediately after realization of capital gains or losses. Specifically, before the year-end settlement, the evolution processes of the money market account $\tilde{x}_t$ and the accumulated taxes to be paid (ξ_t, η_t, ζ_t) are given as follows:

$$
\begin{aligned}
d\tilde{x}_t &= (r\tilde{x}_t + \delta y_t - c_t)dt - (1+\theta)dL_t + (1-\alpha)y_{t-}dM_t, \\
d\xi_t &= \tau\big[(1-l)((1-\alpha)y_{t-} - k_{t-}) + l((1-\alpha)y_{t-} - k_{t-})^+\big]dM_t, \\
d\eta_t &= \tau_i r\tilde{x}_t dt, \\
d\zeta_t &= \tau_d \delta y_t dt,
\end{aligned}
$$

where ξ_t, η_t, and ζ_t are accumulated taxes to be paid on realized stock gains, interest, and dividends, respectively. Let $x_t = \tilde{x}_t - g(t;1)(\xi_t + \eta_t + \zeta_t)$ be the discounted after-tax value of the money market account. It follows that

$$
\begin{aligned}
dx_t &= d\tilde{x}_t - g(t;1)d\xi_t - g(t;1)d\eta_t - g(t;1)d\zeta_t - g'(t;1)(\xi_t + \eta_t + \zeta_t)dt \\
&= (r\tilde{x}_t + \delta y_t - c_t)dt - (1+\theta)dL_t + (1-\alpha)y_{t-}dM_t \\
&\quad -g(t;1)\tau\big[(1-l)((1-\alpha)y_{t-} - k_{t-}) + l((1-\alpha)y_{t-} - k_{t-})^+\big]dM_t \\
&\quad -g(t;1)\tau_i r\tilde{x}_t dt - g(t;1)\tau_d\delta y_t dt - g'(t;1)(\xi_t + \eta_t + \zeta_t)dt \\
&= \Big[r\big(1 - \tau_i g(t;1)\big)x_t + \delta(1 - \tau_d g(t;1))y_t - c_t\Big]dt - (1+\theta)dL_t + f(t, 0, y_{t-}, k_{t-}; l, \lambda)dM_t \\
&\quad -\big[g'(t;1) - r\big(1 - \tau_i g(t;1)\big)g(t;1)\big](\xi_t + \eta_t + \zeta_t)dt.
\end{aligned}
$$

Note that

$$g'(t;1) - r\big(1 - \tau_i g(t;1)\big)g(t;1) = 0,$$

which leads to dynamics (1) for $\lambda = 1$.

A.2 Proof of Proposition 1

It is apparent that the value function is independent of time when $\lambda = 0$. Now, let us consider the case when $\lambda = 1$. Clearly, $\bar{\mathscr{S}_t} = \bar{\mathscr{S}}_{t+1}$. Starting from $s = t+1$, given that $(c_s, L_s, M_s) \in \mathscr{A}_{t+1}(x, y, k)$, we denote by (x_s, y_s, k_s) the solution of (1)-(3). Then we have

$$\begin{aligned} V(t+1, x, y, k) &= \max_{(c_s, L_s, M_s)\in\mathscr{A}_{t+1}(x,y,k)} \mathbb{E}_{t+1}^{x,y,k}\left[\int_{t+1}^{\infty} e^{-\beta(s-t-1)}U(c_s)ds\right] \\ &= \max_{(\tilde{c}_v, \tilde{L}_v, \tilde{M}_v)\in\mathscr{A}_t(x,y,k)} \mathbb{E}_{t}^{x,y,k}\left[\int_{t}^{\infty} e^{-\beta(v-t)}U(\tilde{c}_v)dv\right] \\ &= V(t, x, y, k). \end{aligned}$$

Here the second equality follows from a change of variable $v = s-1$ and the time translation $(\tilde{c}_v, \tilde{L}_v, \tilde{M}_v) = (c_{v+1}, L_{v+1}, M_{v+1})$ and $(\tilde{x}_v, \tilde{y}_v, \tilde{k}_v) = (x_{v+1}, y_{v+1}, k_{v+1})$.

A.3 Proof of Proposition 2

Denote liquidated wealth by $W_t \equiv x_t + (1-\tau)y_t + \tau k_t$. In the absence of transaction costs, the proof is similar to that in Ben Tahar, Soner, and Touzi (2010), where the key step is to consider the dynamics of W_t:

$$dW_t = [((1-\tau)\mu + (1-\tau_d)\delta)y_t - c_t]dt + (1-\tau)\sigma y_t dB_t.$$

Now, let us look at the transaction costs case. Let $\hat{x}_t = x_t + \tau k_t$ and $\hat{y}_t = (1-\tau)y_t$. It is easy to obtain (11)-(12) by the dynamics (1)-(3), where $\hat{L} = (1-\tau)L$. Furthermore, the no-bankruptcy constraint reduces to $\hat{x}_t + (1-\alpha)\hat{y}_t \geq 0$. The equivalence then follows.

A.4 The CEWL incurred by the optimal strategy assuming the IT rule in the market with the YT rule

We have seen that the optimal strategy with the YT rule is significantly different from the strategy that employs the IT rule. We now compute the certainty equivalent wealth loss Δ (CEWL, as a fraction of the initial wealth) incurred by the optimal strategy with the IT rule (i.e., the IT strategy) under the market of implementing the YT rule (i.e. the YT market). More specifically, let $V(t,x,y,k)$ be the value function under the YT market, and $V^{IT}(t,x,y,k)$ be the value function associated with the IT strategy under the YT market. The CEWL Δ is then defined to solve

$$V^{IT}(0,1,0,0) = V(0,1-\Delta,0,0).$$

Table A.1 reports the CEWL incurred by the IT strategy for different levels of tax rate τ with and without transaction costs in the FR case and in the FC case, respectively. It can be seen that the CEWL is marginal, which is consistent with our previous simulation results that the overall tax payments or rebates do not change to a significant degree when we switch from the IT rule to the YT rule, despite the fact that their optimal tax-timing strategies differ significantly.

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Table A.1: CEWL
incurred by the optimal strategy assuming the IT rule when the YT rule is in place

This table reports the CEWL incurred by the optimal strategy assuming the IT rule when the YT rule is in place. Parameter values: $r = 0.03$, $\delta = 0.02$, $\mu = 0.07$, $\sigma = 0.2$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = \tau_d = 0.35$, and $\alpha = \theta$.

	The FR case		**The FC case**	
τ	$\theta = 0$	$\theta = 0.005$	$\theta = 0$	$\theta = 0.005$
0.15	0.021%	0.027%	0.010%	0.010%
0.20	0.033%	0.040%	0.015%	0.015%
0.25	0.046%	0.054%	0.021%	0.020%
0.30	0.067%	0.072%	0.025%	0.026%
0.35	0.083%	0.080%	0.031%	0.032%

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