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ABSTRACT

The well-documented nonparticipation in the stock market by many households and the highly negative correlation between stock and housing investment are puzzling. We show that stock and housing markets are cointegrated, and thus households significantly increase housing expenditure, reduce stock investment, and may choose nonparticipation in the stock market at all if they face short-sale constraints. Our model can thus potentially help explain both the puzzle of stock market nonparticipation and the puzzle of the highly negative correlation between stock and housing investment. We also show some empirical evidence that is supportive of the model's main implications.

JEL classification: E21, G11, G50

Keywords: Nonparticipation, cointegration, housing, stock investment

1 Introduction

Only a small fraction of households participate, directly or indirectly, in the stock market. For example, in the United States, only 43% of households own stocks either directly or indirectly (e.g., through retirement plans), while in India, the number is a mere 8%. This limited participation is puzzling because standard models of lifetime consumption and portfolio choice predict that all households, no matter how risk-averse they are or how little wealth they have, should invest in stocks (Samuelson, 1969; Merton, 1969, 1971; Arrow, 1971). Another piece of empirical evidence is the highly negative correlation between housing investment and stock investment across countries and across time. For example, in 2015, the cross-country correlation between housing and stock investments is about -0.59 among 17 countries, including developed countries like the United States and the United Kingdom, and developing countries like China and India, and the cross-time correlation is about -0.71 in the United States (see Appendix for details). This is also puzzling, because the contemporaneous correlation between stock and housing prices is low (about 0.07), and the standard theories predict low correlations between housing investment and stock investment. In this study, we show that even though stock and housing markets have a low contemporaneous correlation, they are significantly cointegrated and this cointegration can help explain both of these puzzles. We also show some empirical evidence that is supportive of the main prediction of our model.

More specifically, we consider the optimal consumption and portfolio choice problem of a household in a continuous-time setting with a risk-free asset, a stock, and two consumption goods: a perishable good and housing service, subject to short-sale constraints on stock and housing investment. Unlike the existing literature, we first show that the stock and housing markets are significantly cointegrated and then study the impact of this cointegration on the optimal investment and consumption policy. Calibrated to the U.S. data, our model shows that the presence of cointegration between stock and housing markets significantly affects households' investment and consumption decisions. In particular, households may choose not to participate in the stock market even when there is no participation cost and the expected excess return on the stock is highly positive and significantly greater than that on the housing. In addition, the participation cost needed for households to never participate in the stock market is significantly smaller than without cointegration. Moreover, even when households do participate in the stock market, the investment amount is significantly reduced because of the cointegration. Furthermore, we find that the stock and housing investments are highly negatively correlated, even when the stock price and house price are independent (and thus standard theories predict zero correlation between stock investment and housing investment). These results are robust to the consideration of the option of renting a house (instead of owning one) and the high illiquidity in the housing market. Our model can thus potentially help explain the significant nonparticipation in stock markets and the highly negative correlation between stock investment and housing investment.

The main intuition is as follows. Even though the contemporaneous correlation between the stock and housing returns is close to zero, the presence of cointegration results in a significant and positive long-run correlation between the stock and housing markets. For example, the correlation between the 5-year stock and housing returns equals 0.2841 and the correlation between the 10-year stock and housing returns is as high as 0.4589.[1] Therefore, there is a strong substitution effect between the housing and stock investments if a household's investment horizon is long (e.g., 10 years). It is this substitution effect that drives our main results. For example, when the conditional expected return of housing is high relative to that of the stock,[2] households optimally borrow in the risk-free market to increase the size of their house.[3] In these states, in addition to borrowing in the bond market, households would also like to short sell the stock to finance the purchase of an even bigger house. However, due to the short-sale constraints, the best households can do is to stay away from the stock market. This is why households may choose nonparticipation in the stock market even if the stock market's unconditional expected return is much greater than that in the housing market and there is no participation cost. In addition, even when households do participate in the stock market, they invest less than in the case without cointegration, because owning a house already exposes a household to some stock market risk in the long run. When there is a participation cost, because of the indirect stock market exposure from investing in a house (due to the long-run correlation), the critical participation cost above which households choose never to participate in the stock market is much smaller than when there is no cointegration. The highly negative correlation between stock investment and house investment implied by our model also follows from this long-run substitution effect of housing investment for stock investment. In addition, if housing is also a consumption good, then it has a dual role: consumption and investment. This dual role magnifies the demand for housing and reduces stock investment further. Allowing a housing rental market may make our results even stronger, because with access to the rental market, households may optimally choose to buy even bigger houses (further reducing

[1] Modeling the driving force behind the empirical evidence that the correlation between the stock and housing markets increases with the horizon is out of this paper's scope. We suspect this horizon dependent correlation could be consistent with the existence of common factors that affect both the stock and housing markets (e.g., underlying production technology), whose effect is confounded by short-term noises, and thus appears to be statistically small in the short-run. However, the effect becomes statistically and economically significant in the long run after the noises are averaged out.

[2] See Section A.3 for empirical evidence for the time-varying conditional expected return of the housing investment relative to the stock investment.

[3] This is consistent with Fischer and Stamos (2013) and Corradin, Fillat, and Vergara-Alert (2014). Fischer and Stamos (2013) show that the households choose a higher housing-to-net-worth ratio in good states of housing market cycles (Table 3 and Figure 1). Corradin, Fillat, and Vergara-Alert (2014) show that the housing portfolio share immediately after moving to a more valuable house is higher during periods of high expected growth in house prices (Figure 4, Table 3, and Table 7).

stock investment) and rent out part of the houses to finance the purchae. This way, households can benefit more when the conditional expected return of housing is high relative to that of stocks.[4] The incorporation of housing market illiquidity does not change our main results either and can even enhance them. This is because, with illiquidity in the housing market, households stay in the same houses for a longer period of time, and for cointegrated processes, the correlation increases with duration. As a result, the substitution effect of housing investment for stock investment increases.

To the best of our knowledge, although various types of cointegration between the stock and housing markets have been found in the existing literature (see, e.g., Anoruo and Braha, 2008; Tsai, Lee, and Chiang, 2012), this paper is the first to study how this cointegration affects household investment behavior and can help explain the puzzle of non-/limited participation in stock markets as well as the puzzle of the highly negative correlation between stock and housing investment. In addition, unlike the existing literature on the cointegration tests for the two markets, we are the first to use the Johansen trace test to establish cointegration in the form of the stationarity of the log of the ratio of the housing price to the stock price raised to an empirically estimated power.

The main prediction of our model is that as the degree of cointegration between housing and stock markets increases, stock investment decreases and stock market nonparticipation increases. To see if this prediction has empirical support, we utilize the U.S. cross-state variations of the degree of conintegration, stock investment, and stock market nonparticipation to examine the relations among the three. Using data from the Panel Study of Income Dynamics (PSID) at the family level in the 2015 and 2017 waves, we calculate the average value of equity in stocks, the average ratio of financial wealth invested in stocks, and the proportion of interviewed families that do not invest in stocks for each state. We find that, consistent with the model prediction, as the degree of cointegration increases, stock investment decreases and nonparticipation in the stock market increases.

In the existing literature, there are several explanations for the nonparticipation and limited participation puzzle. Benzoni, Collin-Dufresne, and Goldstein (2007) consider the impact of the cointegration between labor income and stock market return. They find that because of the cointegration, investors with labor income invest less in the stock market. Despite the different economic contexts, the substitution effect between stock investment and housing in our model is qualitatively similar to that between labor income and stock investment in their model. Vissing-Jørgensen (2002) shows that a moderate participation cost can explain half of the nonparticipation observed in the data. Cocco (2005) finds that housing crowds out stockholdings, which, together with a sizable stock market entry cost, can explain stock market nonparticipation early in life. Yao and Zhang

[4]Even when households cannot afford to buy a house and thus have to rent, our main results still hold as long as they can invest in the housing market through securities such as the Case-Shiller House Index futures, because the driving force behind our main results is the substitution effect between the stock market and housing market investment, which exists regardless of homeownership.

(2005) examine the substitution and diversification effect of equity investment through an optimal dynamic portfolio decision model for households that acquire housing services from either renting or owning a house. They predict that housing investment has a negative effect on stock market participation. Kraft, Munk, and Wagner (2017) propose a rich life-cycle model of household decisions. After considering housing habit, they obtain that stock investments are low or zero for many young agents and then gradually increase as they age. Linnainmaa (2005) argues that short-sale constraints combined with learning can generate nonparticipation even when the constraints are not binding at present. Ambiguity aversion, disappointment aversion, and behavioral, cognitive and psychological constraints are also offered as possible explanations for the nonparticipation puzzle (e.g., Epstein and Schneider, 2008; Cao, Wang, and Zhang, 2005; Ang, Bekaert, and Liu, 2005; Andersen and Nielsen, 2011). Unlike our paper, all these studies ignore the cointegration between the stock and the housing markets. Our model complements these extant theories and may strengthen their explanatory power. For example, our model suggests that the participation cost in Vissing-Jørgensen (2002) and ambiguity aversion in Cao, Wang, and Zhang (2005) required to explain nonparticipation would be significantly smaller if cointegration were incorporated.

Our paper also relates to recent papers on housing decisions. Hemert (2010) investigates household interest rate risk management with a life-cycle asset allocation model that includes mortgage and bond portfolio choice and finds some hedge between housing and interest rate. Fischer and Stamos (2013) set up a regime-switching model with slow-moving time variation in expected housing returns and find that homeownership rates and the share of net worth in a home increase in good states of housing market cycles. Corradin, Fillat, and Vergara-Alert (2014) show that higher expected growth rates in house prices cause house (stock) investment to increase (decrease), but stock investment is still significant even with a high risk aversion.

As for the puzzle concerning the highly negative correlation between stock and housing ownership/investment, although some studies (e.g., Cocco, 2005; Yao and Zhang, 2005) imply a negative relationship, no extant studies have shown whether the magnitudes of the correlations in their models can be as large as those observed in data.

The rest of the paper is organized as follows. Section 2 describes the benchmark cointegration model. Section 3 shows that the stock and housing markets are cointegrated and provides an estimation of cointegration parameter values. In Section 4, we quantitatively illustrate that the cointegration between the stock and housing markets leads to non-/limited participation in the stock market and a highly negative correlation between stock and housing investment. Section 5 demonstrates the robustness of our results to the option of renting and to the presence of house illiquidity. In Section 6, we provide some empirical evidence that is supportive of the predictions of our model. Section 7 concludes the paper. Empirical facts on nonparticipation and correlations, some Hamilton-Jacobi-Bellman (HJB) equations, and all the proofs are provided in the Appendix.

2 The Model

We consider a continuous-time model where a small household (i.e., with no price impact) maximizes its expected utility from consuming a perishable consumption good and possibly consuming a house's service flow. In addition to trading houses and the perishable consumption good in the goods markets, the household can also trade a risky stock and a risk-free bond in the financial market without any transaction costs.

2.1 Financial markets

The bond grows at a constant risk-free rate r. The stock's price S_t evolves according to the following dynamics:[5]

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dB_{St}, \tag{2.1}$$

where $\mu_S > r$ is a constant representing the instantaneous expected return, σ_S is a constant representing the instantaneous volatility of a stock return, and B_{St} is a one-dimensional standard Brownian motion.

2.2 The housing market

To simplify the analysis, we start by assuming that by selling, buying, remodelling, and expanding, the household can continuously adjust the house size without transaction costs.[6] Let I_t denote the national housing price index per square footage at time t. Unlike the existing literature, we allow the stock and housing markets to be cointegrated. More specifically, in a similar spirit to Benzoni, Collin-Dufresne, and Goldstein (2007), we assume that the following log-ratio denoted by

$$R_t = \log I_t - \lambda \log S_t \tag{2.2}$$

for some positive constant λ follows a mean-reverting process

$$dR_t = k(\bar{R} - R_t)dt + \sigma_I dB_{It} - \nu_S dB_{St}, \tag{2.3}$$

where the constant $k \geq 0$ measures the degree of cointegration, $\bar{R}$ denotes the long-term mean, σ_I and ν_S are conditional volatilities, and B_{It} is another standard Brownian motion reflecting the

[5]Like Benzoni, Collin-Dufresne, and Goldstein (2007), we can start with a dividend process and a cointegrated housing service process and specify a pricing kernel to generate the stock price process and the cointegrated housing process described later. The derivation is omitted here to save space, but is available from the authors.

[6]When houses are indivisible and buying/selling a house incurs a transaction cost, as in Grossman and Laroque (1990), the household's problem becomes much more complicated; this scenario is considered in Section 5.2 to show the robustness of our results.

uncertainty in the aggregate housing price index and is independent of B_{St}. In the next section, we show that R_t for some empirically estimated value of λ is indeed mean-reverting. Our specification (2.3) for the log-ratio R_t reflects a long-run cointegration between the stock and housing markets, because it implies that the log-ratio of the house price index to the stock price raised to the power of λ tends to the long-run mean $\bar{R}$ as time passes. When $R_t > \bar{R}$, the house price index tends to decrease over time relative to the stock price, whereas when $R_t < \bar{R}$, the opposite is true.

By (2.2), we have $I_t = e^{R_t} S_t^{\lambda}$, and hence

$$\frac{dI_t}{I_t} = \mu_I(R_t)dt + \sigma_I dB_{It} + (\lambda\sigma_S - \nu_S)dB_{St}, \tag{2.4}$$

where

$$\mu_I(R) = \mu_{I0} + k(\bar{R} - R), \;\; \mu_{I0} = \lambda\mu_S + \frac{1}{2}\sigma_I^2 + \frac{1}{2}\nu_S^2 + \frac{1}{2}\lambda(\lambda-1)\sigma_S^2 - \lambda\sigma_S\nu_S. \tag{2.5}$$

In this paper, we focus on the analysis of the effect of the cointegration between the housing and stock markets. In addition, as shown in the existing literature and our later analysis in Section 3, the contemporaneous correlation between housing and stock returns is almost zero. Accordingly, we will simply assume that the contemporaneous correlation between housing and stock returns is zero, i.e., $\lambda\sigma_S = \nu_S$.[7] The focus of cointegration between stock and housing markets distinguishes our model from others that ignore such cointegration (see e.g., Fischer and Stamos, 2013; Corradin, Fillat, and Vergara-Alert, 2014).

Note that with $\lambda\sigma_S = \nu_S$, although the contemporaneous correlation between housing and stock returns is zero, the housing and stock markets are linked. For example, after a positive shock in B_{St} to the stock return, the log-ratio R_t decreases (equation (2.3)), which in turn increases the conditional expected return of housing $\mu_I(R_t)$ (equation (2.5)). It is this cointegration that drives our main results.

We assume that there is a continuum of households, and each household can buy and sell houses in its local housing market. For household i, the local housing price $H_{it} = I_t\epsilon_{it}$, where ϵ_{it} represents idiosyncratic risk faced by household i and follows

$$\frac{d\epsilon_{it}}{\epsilon_{it}} = \mu_{\epsilon i}dt + \sigma_{\epsilon_i} dB_{it}, \tag{2.6}$$

where $\mu_{\epsilon i}$ and $\sigma_{\epsilon i}$ are constants and B_{it} is a Brownian motion that is independent across i and of all other risks. This implies that

$$\frac{dH_{it}}{H_{it}} = \mu_H(R_t)dt + \sigma_I dB_{It} + \sigma_{\epsilon i} dB_{it}, \tag{2.7}$$

[7]Increasing the correlation would make our results even stronger.

where $\mu_H(R) = \mu_I(R) + \mu_{\epsilon i} = \mu_{H0} + k(\bar{R} - R)$ with $\mu_{H0} = \mu_{I0} + \mu_{\epsilon i}$.

2.3 Preferences

A household derives utility not only from the perishable consumption good that serves as the numeraire but also possibly from the housing service flow that is proportional to the house size. Thus, unlike financial assets, in addition to the role of an investment vehicle, a house may also directly contribute to utility. We assume that the service flow from a house is proportional to the house size and equal to αA_t, where $\alpha > 0$ is a constant and set to 1 without loss of generality. Following the existing literature (see e.g. Damgaard, Fuglsbjerg, and Munk, 2003; Cocco, Gomes, and Maenhout, 2005; Yao and Zhang, 2005; Kraft and Munk, 2011; Fischer and Stamos, 2013; Corradin, Fillat, and Vergara-Alert, 2014), we assume that the household's preferences over the housing service flow and nonhousing goods take the following nonseparable Cobb-Douglas utility form:

$$U(C, A) = \frac{1}{1-\gamma}\left(C^{1-\theta} A^{\theta}\right)^{1-\gamma},$$

where C represents the perishable good consumption, $\theta \geq 0$ measures the preference for housing, and $\gamma > 0$ is the constant relative risk aversion coefficient.

2.4 The household's optimization problem

Let W_t be the household's total wealth in bonds, stocks, and housing, measured in units of the perishable consumption good at time t, and ζ_t, h_t, and c_t denote the fraction of the total wealth W_t in stock, housing, and perishable goods, respectively. According to equations (2.1) and (2.7), W_t satisfies the following stochastic differential equation[8]

$$\frac{dW_t}{W_t} = [r - c_t + \zeta_t(\mu_S - r) + h_t(\mu_H(R_t) - \delta - r)]\, dt + \sigma_S \zeta_t dB_{St} + \sigma_I h_t dB_{It} + \sigma_{\varepsilon i} h_t dB_{it}, \quad (2.8)$$

where δ is the depreciation rate of housing.

Following the existing literature (e.g., Gomes and Michaelides, 2005; Cocco, Gomes, and Maenhout, 2005; Polkovnichenko, 2007; Munk and Sørensen, 2010; Wachter and Yogo, 2010; Lynch and Tan, 2011; Flavin and Yamashita, 2011), we assume that the household cannot short sell stock or houses, that is, $\zeta_t \geq 0$ and $h_t \geq 0$.[9] However, the household can borrow against the house, up to a fraction $(1 - l)$ of the current value of housing, i.e., $\zeta_t + l h_t \leq 1$, where $l \in (0, 1]$ is a constant, representing the maximum leverage allowed for house purchases.

[8]In an earlier version of the paper, we also solved a model with stochastic labor income and obtained the same qualitative results. This analysis is not reported in this version to save space, but is available from the authors.

[9]Note that the optimal c_t must be strictly positive because of the utility function form.

The household chooses the perishable consumption fraction c_t, the stock weight ζ_t, and the housing weight h_t to maximize the expected utility from consumption of the perishable good and the housing service from time 0 to the first jump time $\mathcal{T}$ of an independent Poisson process with intensity δ_M, which represents the mortality rate of the household.[10] Let $\mathcal{A}$ denote the set of all admissible strategies (c_t, ζ_t, h_t), i.e., the strategies that satisfy the budget constraint (2.8), the short-sale constraint $\zeta_t \geq 0$ and $h \geq 0$, and the limited borrowing constraint $\zeta_t + lh_t \leq 1$, for given processes (2.8), (2.7), and (2.3). We define the value function as (note that $h_t = \frac{A_t H_t}{W_t}$)

$$\begin{aligned}
\Psi(W, H, R) : &= \max_{(c_t,\zeta_t,h_t)\in\mathcal{A}} E\left[\int_0^{\mathcal{T}} e^{-\beta t} \frac{\left({c_t}^{1-\theta}(h_t/H_t)^{\theta} W_t\right)^{1-\gamma}}{1-\gamma} dt\right] \\
&= \max_{(c_t,\zeta_t,h_t)\in\mathcal{A}} E\left[\int_0^{\infty} e^{-(\beta+\delta_M)t} \frac{\left({c_t}^{1-\theta}(h_t/H_t)^{\theta} W_t\right)^{1-\gamma}}{1-\gamma} dt\right], \qquad (2.9)
\end{aligned}$$

which satisfies the following HJB equation:

$$\begin{aligned}
\max_{c,\zeta,h\geq 0;\ \zeta+lh\leq 1} \Bigg\{ &\frac{1}{2}[\sigma_S^2\zeta^2 + (\sigma_I^2 + \sigma_{\varepsilon i}^2)h^2]W^2\Psi_{WW} + \frac{1}{2}(\sigma_I^2 + \sigma_{\varepsilon i}^2)H^2\Psi_{HH} + \frac{1}{2}(\lambda^2\sigma_S^2 + \sigma_I^2)\Psi_{RR} \\
&+ (\sigma_I^2 h - \lambda\sigma_S^2\zeta)W\Psi_{WR} + (\sigma_I^2 + \sigma_{\varepsilon i}^2)HhW\Psi_{WH} + \sigma_I^2 H\Psi_{HR} \\
&+ [r - c + (\mu_S - r)\zeta + (\mu_H(R) - \delta - r)h]W\Psi_W + \mu_H(R)H\Psi_H \\
&+ k(\bar{R} - R)\Psi_R - (\beta + \delta_M)\Psi + \frac{\left(c^{1-\theta}(h/H)^{\theta}W\right)^{1-\gamma}}{1-\gamma} \Bigg\} = 0
\end{aligned}$$

for $W > 0$, $H > 0$, and $R \in \mathbb{R}$.

Using the homogeneity property of the value function, we can reduce the dimensionality of the problem to one by the following transformation:

$$\Psi(W, H, R) = \frac{1}{1-\gamma} W^{1-\gamma} H^{-\theta(1-\gamma)} e^{(1-\gamma)u(R)},$$

[10] The assumption of a random horizon eliminates the time dependence of the optimal strategies. Using a deterministic horizon would not change our qualitative results. In addition, as Liu and Lowenstein (2002) suggest, the optimization problem with a random horizon can be a good approximation for a deterministic horizon when the expected horizon is long.

for some function $u(\cdot)$. It can be shown that the function $u(R)$ satisfies:

$$\max_{c,\,\zeta,\,h\geq 0;\,\zeta+lh\leq 1}\Big\{\frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)[u''+(1-\gamma)u'^2]+\Big[(\sigma_I^2 h-\lambda\sigma_S^2\zeta-\theta\sigma_I^2)(1-\gamma)+k\bar{R}-kR\Big]u'$$
$$-\frac{1}{2}\gamma(\sigma_S^2\zeta^2+(\sigma_I^2+\sigma_{\varepsilon i}^2)h^2)+\frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(\theta(1-\gamma)+1)-(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(1-\gamma)h+r-c \tag{2.10}$$
$$+(\mu_S-r)\zeta+(\mu_H(R)-\delta-r)h-\theta\mu_H(R)-\frac{\beta+\delta_M}{1-\gamma}+\frac{c^{(1-\theta)(1-\gamma)}h^{\theta(1-\gamma)}}{1-\gamma}e^{-(1-\gamma)u}\Big\}=0$$

for $R\in\mathbb{R}$. With this formulation, we simplify the problem to solving (2.10) for $u(R)$. Because of the presence of cointegration (i.e., $k\neq 0$), the choice of the stock weight ζ and the choice of the housing investment weight are jointly determined because both depend on the function $u(R)$ and its derivatives. In the special case where $k=0$, i.e., there is no cointegration between stock and house price, $u(R)$ is a constant and thus the choice of the stock weight ζ and the choice of the housing investment weight h are independent. We have the following lemma in this special case.[11]

Lemma 1. *Suppose $k=0$ and*

$$\beta+\delta_M-(1-\gamma)\Big\{r+\frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(\theta(1-\gamma)+1)-\mu_{H0}\theta+\frac{(\mu_S-r)^2}{2\gamma\sigma_S^2}$$
$$+\frac{(\mu_{H0}-\delta-r-(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(1-\gamma))^2}{2\gamma(\sigma_I^2+\sigma_{\varepsilon i}^2)}\Big\}>0. \tag{2.11}$$

We have $\Psi(W,H,R)\equiv\frac{K}{1-\gamma}W^{1-\gamma}H^{-\theta(1-\gamma)}$, where

$$K = (h^*)^{\theta(1-\gamma)}\eta^{1-p}\Big[-\frac{1}{2}\gamma\sigma_S^2(\zeta^*)^2+(\mu_S-r)\zeta^*-\frac{1}{2}\gamma(\sigma_I^2+\sigma_{\varepsilon i}^2)(h^*)^2+(\mu_{H0}-\delta-r$$
$$-(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(1-\gamma))h^*+\frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(\theta(1-\gamma)+1)+r-\mu_{H0}\theta-\frac{\beta+\delta_M}{1-\gamma}\Big]^{p-1},$$

with $p=(1-\theta)(1-\gamma)$, $\eta=(1-\theta)^{\frac{1}{1-p}}-\frac{(1-\theta)^{\frac{p}{1-p}}}{1-\gamma}$, and (ζ^*,h^*) being the optimal stock and house investment satisfying

$$(\zeta^*,h^*) = \underset{\zeta,\,h\geq 0;\,\zeta+lh\leq 1}{\arg\max}\Big\{-\frac{1}{2}\gamma\sigma_S^2\zeta^2+(\mu_S-r)\zeta-\frac{1}{2}\gamma(\sigma_I^2+\sigma_{\varepsilon i}^2)h^2+(\mu_{H0}-\delta-r$$
$$-(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(1-\gamma))h-K^{-\frac{1}{1-p}}\eta h^{\frac{\theta(1-\gamma)}{1-p}}\Big\}.$$

[11]Similar to Merton's problem (Merton, 1969, 1971), to guarantee the existence of a solution, we impose (2.11), because if this condition fails, then the household can achieve unbounded utility by delaying consumption.

3 Cointegration Test

In this section, we aim to test whether there is cointegration between stock price and housing index and estimate the cointegration degree if there is. The sources of the stock market data and the house price index series are Standard & Poor's and the Case-Shiller Home Price Indices (CSI), respectively, both of which are inflation-adjusted to November 2019 dollars. We use the annual data on December 1 from 1890 to 2017.

Before conducting the test, we first estimate the contemporaneous correlation between the housing index and stock returns in our data set. We find that the correlation between $\log\left(\frac{S_t}{S_{t-1}}\right)$ and $\log\left(\frac{I_t}{I_{t-1}}\right)$ is 7.14%, which is consistent with the findings in the existing literature and leads to our simplifying assumption that the housing price index and the stock price have zero contemporaneous correlation (i.e., $\lambda\sigma_S = \nu_S$).

We use the trace test proposed by Johansen (1988,1991) to examine whether the stock and housing markets are cointegrated, because this test is an improvement over the two-step test proposed by Engle and Granger (1987). The normalized cointegration vector $(1, -\lambda)$ is estimated using Maximum Likelihood Estimation (MLE) methods. The estimates are asymptotically normal and super consistent. Using this method on our data set, we find that the Johansen MLE estimator $\hat{\lambda}^{MLE}$ equals 0.2695. As a result, we set $\lambda = \hat{\lambda}^{MLE} = 0.2695$ in benchmark calibration throughout the paper.[12] The trace test shows that the residual process $R_t = \log I_t - \lambda \log S_t$ follows an AR(1) model:

$$R_{t+\Delta t} = m + \phi R_t + \epsilon_{t+\Delta t}, \tag{3.1}$$

where Δt is the time between adjacent observations with $m = 0.5998$ and $\phi = 0.8180$. Then, we can compare equations (2.3) and (3.1) to imply the speed of the mean-reversion coefficient k, the mean $\bar{R}$, and variance σ_I in equation (2.3).

Figure 1 shows the probability distribution function of $\mu_I(R)$ based on the historic data from 1890 to 2017, which we use to calibrate some default parameter values. From this figure, we find that from 1890 to 2017, the values of $\mu_I(R)$ were concentrated in the interval $(-0.0785, 0.0709)$.

As noted before, even if the stock and housing markets are contemporaneously uncorrelated, the two markets will be correlated for a longer horizon if they are cointegrated. To see if the set of parameter values estimated above is reasonable, we next compute the model-implied long-term correlations between stock and housing index returns for horizons of 1 year, 5 years, and 10 years. We then compare these correlations with the corresponding empirical correlations in the data. Table 1 shows that the model-implied correlations match well with the empirical correlations,

[12] We also conducted the two-step Engle and Granger test. The results are similar. For example, the estimate for λ in this alternative test is 0.2471 and setting $\lambda = 0.2471$ in the benchmark calibration does not significantly change our main results.

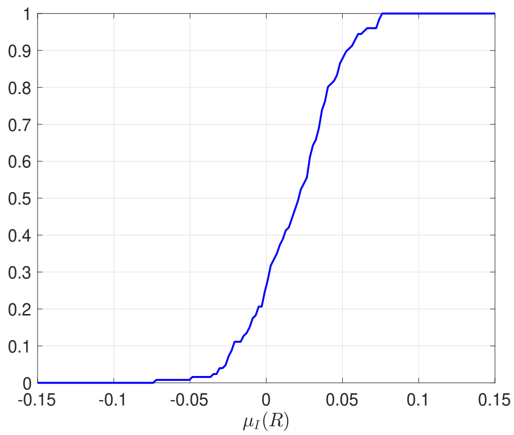


Figure 1: **Probability distribution function of $\mu_I(R_t)$.**

Table 1: **Long-term correlation**

	Correlation between house and stock return		
	1-year	5-year	10-year
Historical data observation	0.0714	0.2841	0.4589
Cointegration model implied	0.0874	0.2807	0.4483

which suggests the estimated model reflects the data reasonably well.

4 Numerical Analysis

4.1 Parameter values

In Table 2, we report the default parameter values for our numerical analysis. We set the inflation-adjusted interest rate at $r = 0.59\%$ based on the five-year real interest rates, the stock risk premium at 2.80% (i.e., the stock return $\mu = 3.39\%$), and the standard deviation of stock return at $\sigma_S = 14.11\%$, according to the estimates of Standard & Poor's 500 index portfolio inflation-adjusted to November 2019 dollars. The coefficient of relative risk aversion is set at $\gamma = 10$ to approximately match the stockholdings relative to financial wealth observed in the PSID and Survey of Income and Program Participation (SIPP) literature. The parameter θ that measures the degree to which the household values housing consumption is set at 0.3 to be consistent with the average share of household housing expenditure in the United States (see, e.g., Corradin, Fillat, and Vergara-Alert, 2014). Households can short bonds to finance homeownership and the minimum housing down payment for homeowners is 20%, which implies that $l = 0.2$. The values of λ, k, $\bar{R}$, and σ_I are estimated in Section 3. We estimate the idiosyncratic risk of local house price by comparing the

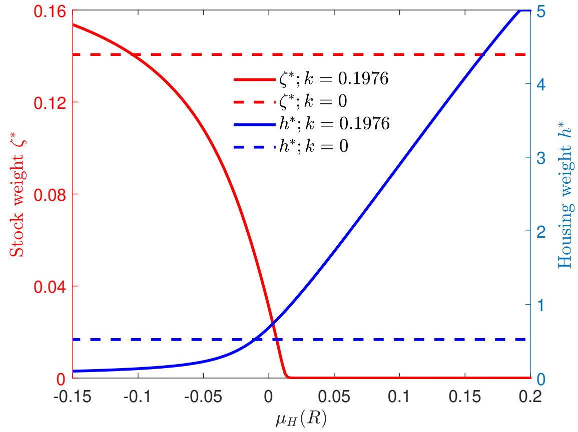
52 states' house price indices to the national house price index. For each state, we assume that the ratio of the annual state house price index divided by the national house price index follows a geometric Brownian motion. Then, we average across states to get the estimated parameters: $\mu_{\varepsilon i} = 0.0056$ and $\sigma_{\varepsilon i} = 0.0316$. One limitation of the state-level idiosyncratic risk is that we can only get the optimal investment and consumption policy of a representative household in a state and cannot have heterogeneity within a state. If we were able to obtain zip code-level housing prices, we would be able to obtain such heterogeneity.

Table 2: **Parameter values used for benchmark calibration**

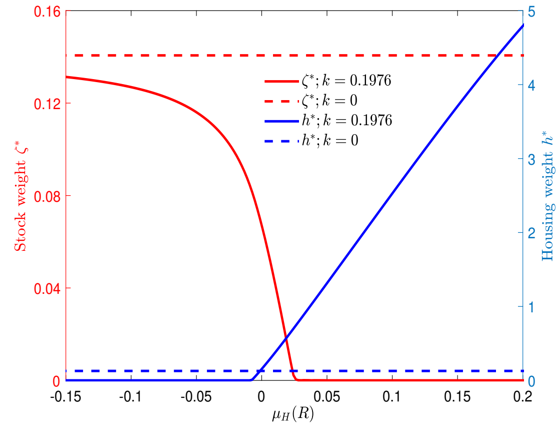
Variable	Symbol	Value
Riskless rate	r	0.0059
Stock expected return	μ_S	0.0339
Stock volatility	σ_S	0.1411
Weight of cointegration	λ	0.2695
Degree of cointegration	k	0.1976
Long-term log-ratio mean	$\bar{R}$	3.2162
Housing index volatility	σ_I	0.0791
Housing index return long-term mean	μ_{I0}	0.0096
Idiosyncratic housing price risk mean	$\mu_{\varepsilon i}$	0.0056
Idiosyncratic housing price risk volatility	$\sigma_{\varepsilon i}$	0.0316
Time discount rate	β	0.0059
Mortality rate	δ_M	0.05
Risk aversion coefficient	γ	10
The preference for housing	θ	0.3
Depreciation rate for housing	δ	0
Housing collateral rate	l	0.2

4.2 Optimal investment and consumption policies

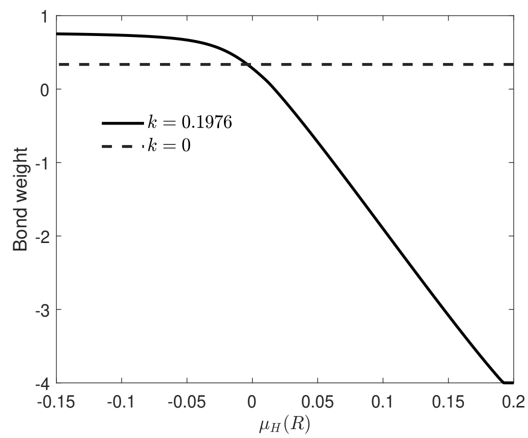
This section demonstrates how cointegration between stock and housing markets affects optimal investment and consumption policies. In particular, we show that, compared with a model that ignores cointegration, households invest significantly less in the stock market and significantly more in housing. With cointegration, they may choose not to participate in the stock market at all, even when there is no participation cost and the unconditional expected return of the stock is greater than that of housing. In addition, because of the cointegration, the stock investment and the housing investment display a strong negative correlation over time. Our model can thus help explain the observed non-/limited participation in stock markets and the strong negative correlation between stock investment and housing investment.



(i) Optimal stock and housing investment



(ii) Optimal stock and housing investment, $\theta = 0$



(iii) Optimal bond investment

Figure 2: **Optimal investment policy.** In Panels (i) and (ii), h^* (the blue-solid line) is the optimal housing weight and the blue-dotted line is the optimal housing weight when $k = 0$. They are measured on the right vertical axis. ζ^* (the red-solid line) is the optimal stock weight and the corresponding red-dotted line is the optimal stock weight when $k = 0$. They are measured on the left vertical axis. Panel (iii) plots the optimal bond weight. Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$.

4.2.1 Correlation between stock and housing investment

The optimal investment in stocks, housing, and bonds are all functions of the conditional expected housing investment return $\mu_H(R)$. Accordingly, in Figure 2, we plot the optimal investment in housing and stocks (Panels (i) and (ii)) and in bonds (Panel (iii)) against $\mu_H(R)$ with and without cointegration. Panel (i) of Figure 2 suggests that when stock investment increases, housing investment decreases, and vice versa. This pattern suggests that, consistent with empirical evidence illustrated in the Appendix, housing investment and stock investment are negatively correlated for each household over time. Intuitively, in the presence of the cointegration effect (i.e., $k > 0$), the expected return of housing is time-varying and stochastic. Although the long-term expected return of housing μ_{H0} is low, the conditional expected return of housing can be high relative to that of stocks in some states (e.g., when the log-ratio R_t is low). Knowing that housing and stock markets are cointegrated and thus correlated in the long term, the household levers up more to increase the house size by borrowing more (as indicated by the corresponding negative bond holdings in Panel (iii)) and decreasing stock investment in these states, consistent with the finding of Fischer and Stamos (2013). When the conditional expected return of housing $\mu_H(R)$ is low, the reverse is true. These changes in the relative conditional expected returns and the cointegration between stock and housing cause the negative correlation between stock investment and housing investment.

One might suspect that the increase in the housing investment and decrease in the stock investment are mainly due to the assumption that housing is not only an investment vehicle, but also a consumption good. Panel (ii) of Figure 2 suggests that there is still a negative correlation between stock investment and housing investment even when the household does not derive utility directly from housing (i.e., when $\theta = 0$). In contrast, if there were no cointegration, then the household would always invest constant and positive fractions of wealth in stock and in housing, as indicated by the dashed lines in Panels (i) and (ii). Thus, without cointegration, the correlation between the fractions of wealth invested in stock and housing would be zero. Therefore, it is not the additional role of housing as a consumption good that drives the negative correlation result. Rather, the key driver is the cointegration between the stock and the housing markets. The existing literature ignores the cointegration and as a result, given the low contemporaneous correlation between the stock and housing returns, it cannot explain the highly negative correlation between stock and housing investment. This contrast with the existing literature suggests the importance of cointegration in helping to solve the negative correlation puzzle.

We next estimate the magnitude of the correlation implied by our model through simulations. For this purpose, similar to Kraft, Munk, and Wagner (2017), we set the initial financial wealth at $\widetilde{W}_0 = 20$ (representing \$20,000), in line with the median net worth and before-tax income statistics for young individuals, as documented in the 2013 Survey of Consumer Finance (SCF). We set the initial housing price at $H_0 = 0.25$ (i.e., \$250 per square foot) and the initial log-ratio R_0 to be its long-term average $\bar{R}$. We then simulate 10,000 paths of the process R_t using equation (2.3), compute stock investment, housing investment, and their correlation for each path, and then average them across all paths. At any point in time in the simulation of a sample path, we keep the realizations of the common risk factors B_{St} and B_{It} the same across all 52 states, but draw the idiosyncratic housing risk factor B_{it} independently across states. We present the average cross-sectional results in Table 3 for various parameter values. Consistent with Figure 2 and the related empirical evidence, we find a highly negative correlation between housing and stock investment, for example, -0.6529 in the base case. When there is no cointegration, however, the correlation is zero, as expected. In addition, the presence of cointegration decreases average stock investment and increases average housing investment. Thus, the presence of cointegration may help explain the puzzle of a highly negative correlation between stock and housing investment. When the household has a greater preference for housing service consumption (i.e., θ is larger), stock investment decreases, housing investment increases, and the correlation between the two becomes more negative. When the household cannot borrow against house equity (e.g., $l = 1$), the housing investment decreases and the stock investment increases. With a lower risk aversion (e.g., $\gamma = 5$), both the housing investment and the stock investment increase. In both cases, the region of constant stock investment shrinks and thus the magnitudes of the correlation increase.[13]

Table 3: **Simulation.** Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$.

Parameters		Stock Investment (ζ^*)			House Investment (h^*)			Investment Correlation		
		Mean	Median	Std	Mean	Median	Std	Mean	Median	Std
Base case	$k = 0$	0.1407	0.1407	0	0.5260	0.5398	0.0537	0	0	0
	$k = 0.1976$	0.0178	0.0089	0.0210	1.1974	1.3000	0.2579	-0.6529	-0.5853	0.1576
$\theta = 0$	$k = 0$	0.1407	0.1407	0	0.1279	0.1437	0.0512	0	0	0
	$k = 0.1976$	0.0381	0.0288	0.0225	0.5719	0.6349	0.1781	-0.5772	-0.4833	0.1899
$\theta = 0.6$	$k = 0$	0.1407	0.1407	0	0.8605	0.8816	0.0584	0	0	0
	$k = 0.1976$	0.0129	0.0039	0.0208	1.5432	1.6530	0.2761	-0.8180	-0.7849	0.0861
$l = 1$	$k = 0$	0.1407	0.1407	0	0.5260	0.5398	0.0537	0	0	0
	$k = 0.1976$	0.0464	0.0412	0.0166	0.7190	0.7441	0.0777	-0.8293	-0.8275	0.0140
$\gamma = 5$	$k = 0$	0.2813	0.2813	0	0.7601	0.7853	0.0996	0	0	0
	$k = 0.1976$	0.0733	0.0573	0.0407	1.5814	1.6939	0.3176	-0.8743	-0.8513	0.0680

4.2.2 Stock market nonparticipation

Panel (i) of Figure 2 also shows that, even with an adjusted stock risk premium at 2.81% and no participation cost for stock investment, a household may choose not to participate in the stock

[13]This is because in the region where the stock investment stays constant, the correlation of the housing investment and the stock investment is zero. When this region shrinks, the overall correlation increases.

market at all, and "underinvest" even when it does choose to participate, which is consistent with the empirically documented non-/limited stock market participation. In particular, when $\mu_H(R) > \mu_H^*$, the threshold value of the conditional expected return of housing for nonparticipation, we have $\zeta_t^* = 0$, i.e., the household optimally chooses not to participate. The probability of nonparticipation in the stock market is equal to

$$P(\mu_H(R) \geq \mu_H^*), \tag{4.1}$$

where $\mu_H(R) = \mu_I(R) + \mu_{\varepsilon i}$ and the the distribution of $\mu_I(R)$ is shown in Figure 1. In Panel (i) of Figure 2, the nonparticipation threshold μ_H^* equals 0.0144, which implies that the probability of nonparticipation equals 0.52. Thus, there is a significant probability of stock market non-particiaption even when the beta of the housing market (with respect to the stock market) is zero and the unconditional expected market risk premium of housing $\mu_{H0} - r$ is less than that of the stock risk premium. The intuition is straightforward. Given the cointegration between stock and housing, the household levers up to increase the house size by borrowing more and decreasing stock investment in the states where the *conditional* expected return of housing is high. In addition, the household would like to short sell the stock if possible to provide funds to further increase the house size. Because of the short-sale constraint, however, the best the household could do is to stop participating in the stock market. When the conditional expected return of housing decreases (equivalently when R_t rises), the household reduces investment in housing and begins to invest in stock for the relatively higher expected return in the stock.

Figure 2 suggests that there is still stock market nonparticipation even when the household does not derive utility directly from housing (i.e., when $\theta = 0$). In contrast, if there were no cointegration, then the household would always invest in stock, as indicated by the dashed lines in Panels (i) and (ii). Therefore, like the negative correlation result, it is not the additional role of housing as a consumption good that drives the nonparticipation result. It is the cointegration between the stock and the housing markets that causes the nonparticipation. In addition, Panel (ii) of Figure 2 also suggests that, consistent with empirical evidence, there is a significant negative correlation between house ownership and stock ownership, i.e., when a household owns a house, it is more likely that it does not own stocks, and vice versa.[14]

The nonparticipation threshold μ_H^* and the probability of nonparticipation depend on the model parameters. For example, Figure 3 shows that when θ increases, housing investment becomes more important and valuable, because in addition to financial returns housing investment also provides

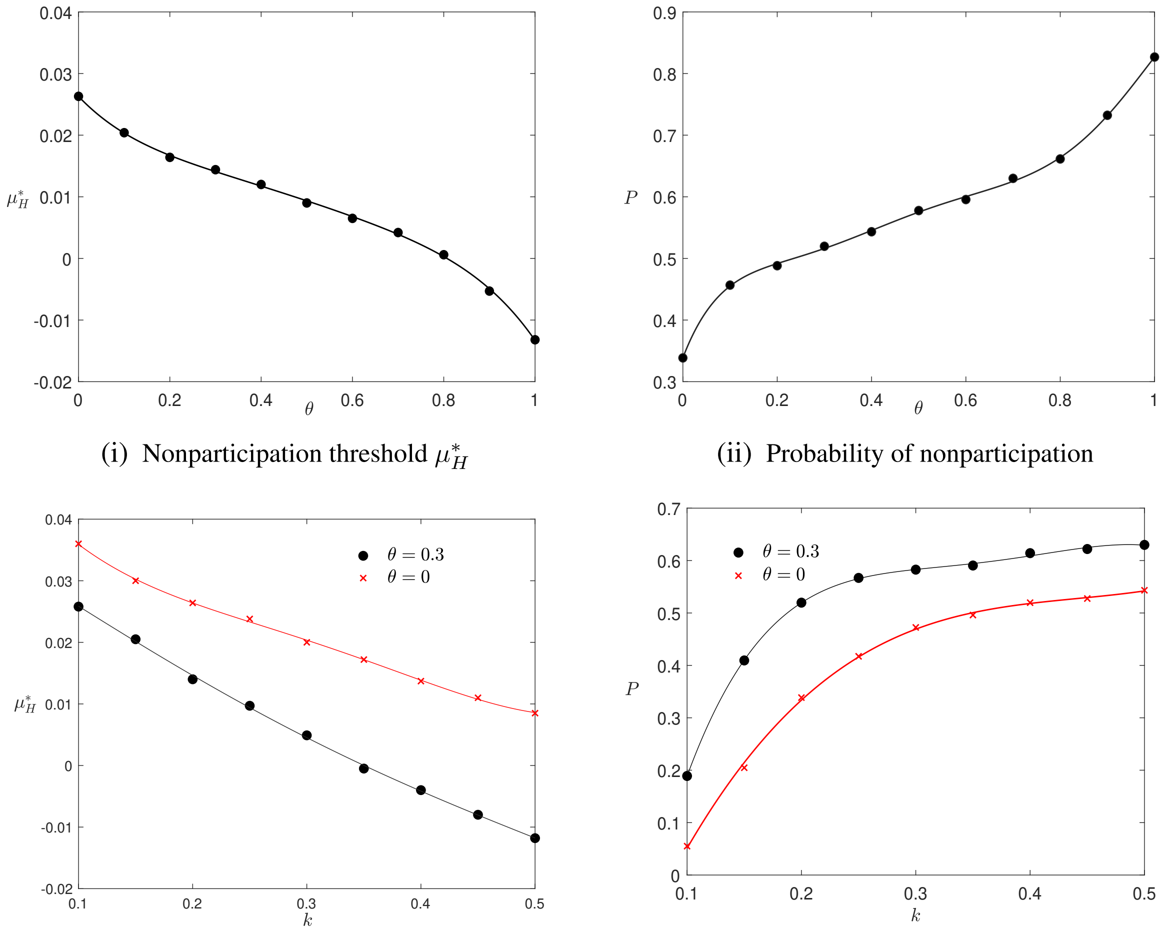
[14]Given the Cobb-Douglas utility function we use, as long as housing is a direct consumption good (i.e., $\theta > 0$), the household always owns a house. The main intuition behind our results suggests that there would be a negative correlation between house ownership and stock ownership even when housing is a direct consumption good if another form of utility function were used such that it was not necessary for the household to always own a house.

higher marginal utility if θ is larger. As a consequence, the nonparticipation threshold μ_H^* moves down, which implies a higher probability of nonparticipation in the stock market. Recall that the parameter k measures the degree of cointegration: if k is larger, the stock price and housing price tend to move closer together. Stronger cointegration enhances the substitution effect of the housing market risk for the stock market risk, and thus the portfolio share of wealth in stocks decreases, leading to a lower nonparticipation threshold μ_H^* and a higher probability of nonparticipation in the stock market. If the household does not get direct utility from housing (i.e., $\theta = 0$), housing is less attractive, thus the nonparticipation threshold μ_H^* is greater and the probability of nonparticipation in the stock market is lower but still significant (e.g., 0.33 when $k = 0.1976$).

In Figure 4, we plot the nonparticipation threshold μ_H^* and the probability of nonparticipation against risk aversion γ and mortality rate δ_M. Figure 4 shows that when the household becomes more risk-averse, the nonparticipation threshold μ_H^* decreases and the probability of nonparticipation increases. This is because the household is less willing to invest in stocks. If housing does not contribute directly to utility (i.e., $\theta = 0$), then the nonparticipation threshold μ_H^* is higher and the stock market nonparticipation probability is lower. The nonparticipation threshold μ_H^* and the probability of nonparticipation also vary with other model parameters, but we find that the effect of changing other parameter values is relatively small. For example, with a higher mortality rate δ_M, the household consumes more perishable goods, which only slightly increases the nonparticipation likelihood, as shown in Panels (iii)–(iv) of Figure 4.

4.2.3 Optimal perishable good consumption

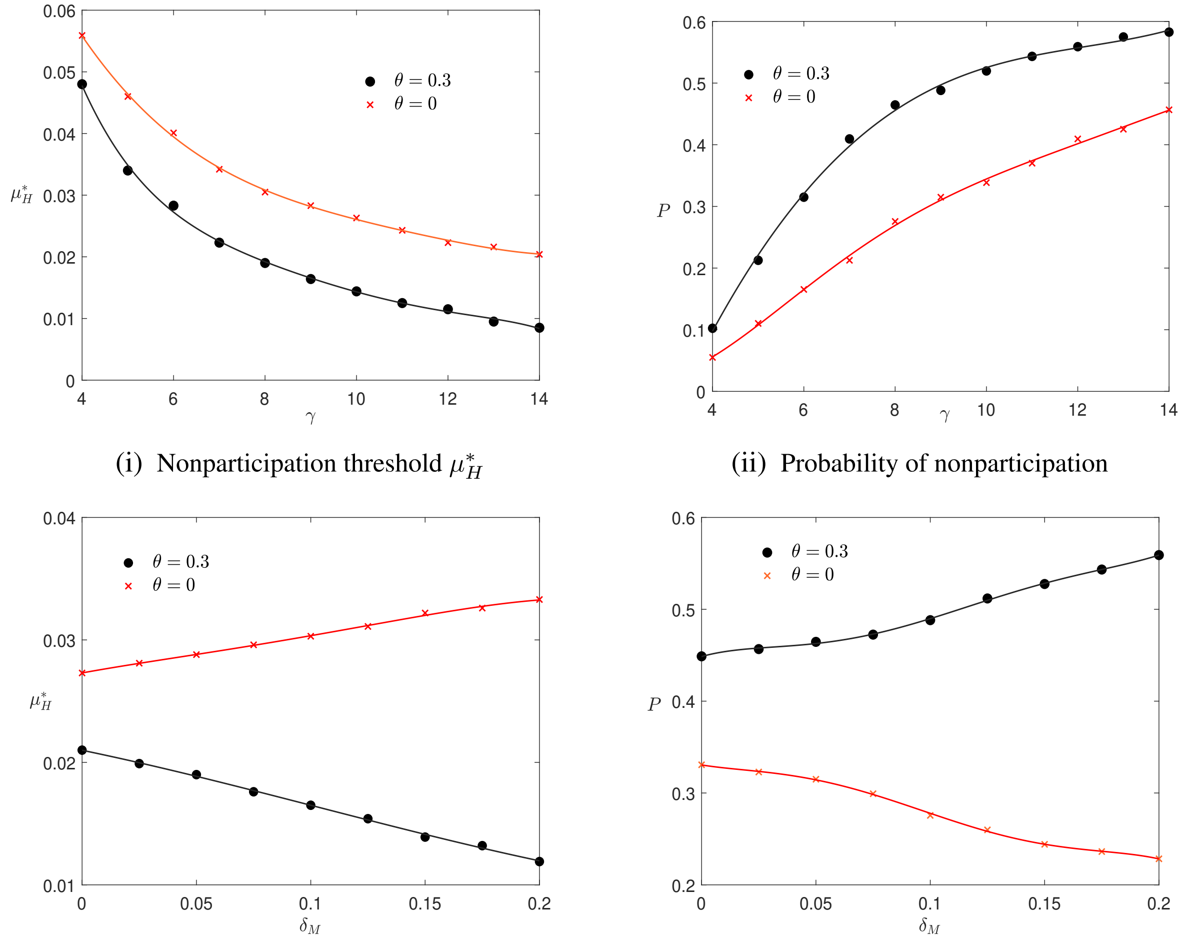
After examining the impact of cointegration on risk-taking in the housing and the stock markets, we next turn to its impact on a household's consumption of the perishable good. To illustrate this impact, we plot the optimal perishable good consumption against the conditional expected return of housing $\mu_H(R)$ with and without cointegration in Figure 5. Figure 5 shows that the fraction of wealth spent on perishable good consumption first decreases and then increases with the conditional expected return of housing $\mu_H(R)$. The intuition is as follows. Changes in the conditional expected return of housing have two opposing effects: the substitution effect and the wealth effect. When the conditional expected return of housing is high, the household invests more in housing and the substitution effect tends to decrease perishable good consumption. On the other hand, because of the greater return from the housing investment, wealth tends to grow faster, which tends to increase perishable good consumption. When the conditional expected return of housing is low, the household invests less in housing and thus the substitution effect tends to increase perishable good consumption. On the other hand, because of the lower expected return from the housing investment, the growth in wealth becomes slower, which tends to decrease perishable good consumption. Therefore, whether perishable good consumption increases or not depends on which effect domi-



(i) Nonparticipation threshold μ_H^* (ii) Probability of nonparticipation

(iii) Nonparticipation threshold μ_H^* (iv) Probability of nonparticipation

Figure 3: **Nonparticipation against θ and k.** When the conditional expected return of housing $\mu_H(R)$ is above μ_H^*, there is nonparticipation in the stock market. The probability of nonparticipation is defined in (4.1). Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$.



(i) Nonparticipation threshold μ_H^* (ii) Probability of nonparticipation

(iii) Nonparticipation threshold μ_H^* (iv) Probability of nonparticipation

Figure 4: **Nonparticipation against γ and δ_M.** When the conditional expected return of housing $\mu_H(R)$ is above μ_H^*, there is nonparticipation in the stock market. The probability of nonparticipation is defined in (4.1). Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$.

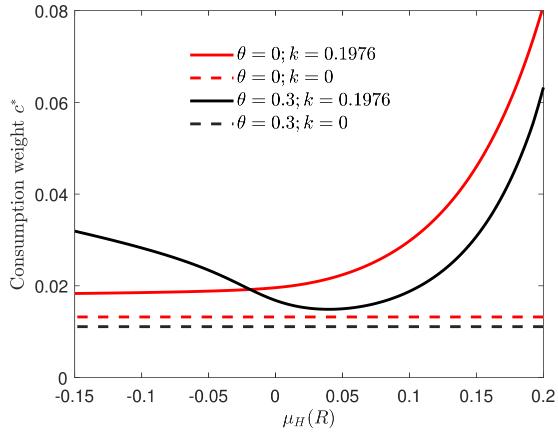


Figure 5: **Optimal perishable good consumption.** c^* is the optimal weight of consumption of perishable goods. Default parameters are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$.

nates. As shown in Figure 5, when the conditional expected return of housing is low, households consume the perishable good at a higher rate, indicating that the substitution effect dominates because the household significantly decreases housing investment and thus the marginal utility from perishable good consumption is higher. In contrast, if the conditional expected return of housing is high, the perishable good consumption rate is also high. This is because the much greater expected return from the housing investment significantly increases the wealth effect, which becomes the dominant effect. This explains the nonmonotonicity of the perishable good consumption in the conditional expected return of housing $\mu_H(R)$.

If housing does not contribute directly to the household's utility (i.e., $\theta = 0$), then perishable good consumption increases with the conditional expected return of housing $\mu_H(R)$. This result is driven by the absence of the substitution effect between housing and the perishable good. As the conditional expected return of housing increases, because there is only the wealth effect, perishable good consumption increases. In addition, when the conditional expected return of housing $\mu_H(R)$ is high, the household consumes more perishable goods compared to the case where housing contributes directly to the household's utility, because the marginal utility of the perishable good consumption is higher with $1 - \theta = 1$ and there is no substitution effect from an increase in housing. However, when the conditional expected return of housing $\mu_H(R)$ is low, the household consumes less perishable goods compared to the case where housing contributes directly to the household's utility. This is because the substitution effect in the case where housing contributes directly to the household's utility increases perishable good consumption when the housing service is low due to the lower conditional expected return of housing $\mu_H(R)$.

4.3 Cost of ignoring cointegration

We have so far shown that cointegration significantly changes a household's investment strategy. But does cointegration make a significant difference in a household's expected utility? In this subsection, we analyze the equivalent wealth loss from ignoring the cointegration between stock and housing markets. If a household ignores the cointegration effect between stock and housing markets, i.e., assumes $k = 0$, the house price index and local house price follow geometric Brownian motions:

$$\begin{aligned} dI_t &= \mu_I^0 I_t dt + \sigma_I^0 I_t dB_{It}, \\ \frac{dH_{it}}{H_{it}} &= (\mu_I^0 + \mu_{\varepsilon i})dt + \sigma_I^0 dB_{It} + \sigma_{\varepsilon i} dB_{it}. \end{aligned} \tag{4.2}$$

Using the home price index on December 1 from 1890 to 2017 inflation-adjusted to November 2019 dollars and assuming no cointegration, we have the following new estimates for the household: $\mu_I^0 = 0.0068$ and $\sigma_I^0 = 0.0711$. We denote by $\Psi^0(W, H, R)$ the value function of the household that adopts the optimal trading policy, incorrectly assuming no cointegration in a market with cointegration. The equivalent wealth loss ΔW of cointegration can be defined as

$$\Psi(W - \Delta W, H, \bar{R}) = \Psi^0(W, H, \bar{R}), \tag{4.3}$$

where Ψ is the value function of a model that correctly incorporates the cointegration in (2.9). We plot the equivalent wealth loss as a fraction of the initial wealth from ignoring the cointegration against the preference for housing parameter θ in Figure 6. Figure 6 shows that ignoring the cointegration can be costly to a household. For example, at $\theta = 0.3$, the equivalent wealth loss is about 38% of the initial wealth.

4.4 Stock market participation cost

So far, we have shown that a household may choose not to participate in the stock market in some states of the world (i.e., when the conditional expected return of housing $\mu_H(R)$ is high) and that such a decision is independent of the wealth level. In practice, however, some households never participate in the stock market and the nonparticipation rate decreases with the wealth level. To help explain this empirical evidence, we now extend our model to include a one-time, fixed participation cost (e.g., cost of attention, stress, information processing) for participation in the stock market. More specifically, to participate in the stock market from time 0 to time $\mathcal{T}$, a household must pay a one-time cost of η at time 0. For a given level of participation cost η, we can solve for the critical wealth level $\bar{W}$ below which a household will choose never to participate. Note that

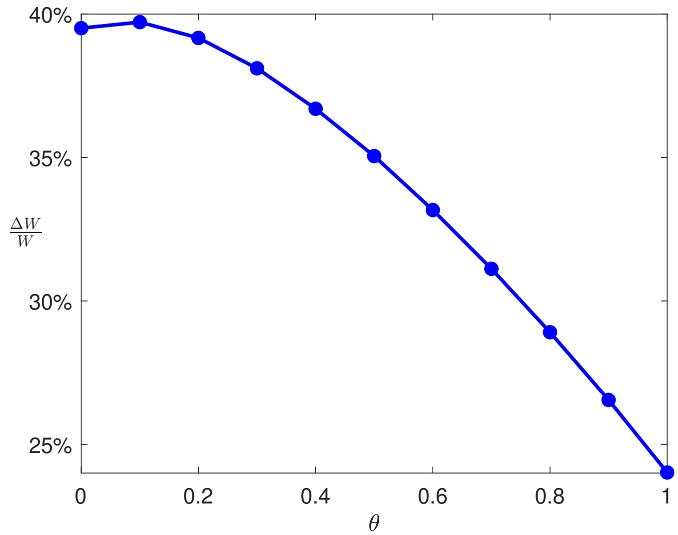


Figure 6: **Equivalent wealth loss from ignoring cointegration.** The blue line depicts the relative equivalent loss $\Delta W/W$ under different θ. Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$. For the model without cointegration: $\mu_{I0}^0 = 0.0068$ and $\sigma_I^0 = 0.0711$.

even after a household pays the participation cost η at time 0, the household may still choose not to participate when the conditional expected return of housing is low, as we have shown above.

Let $\bar{W}$ be the critical wealth level below which a household never participates in the stock market. For given η, H_0 and R_0, the critical wealth level $\bar{W}$ at time 0 then solves

$$\Psi(\bar{W} - \eta, H_0, R_0) = \Psi_0(\bar{W}, H_0, R_0),$$

where $\Psi_0(W, H, R)$ is the value function if a household is prohibited from ever investing in the stock market. Because of homogeneity, the solution $\bar{W}$ is independent of H_0 and is only a function of η and R_0. We can then compute the minimum value of $\bar{W}(\eta, R_0)$ across all R_0, i.e.,

$$\bar{W}^*(\eta) = \inf_{R_0} \bar{W}(\eta, R_0).$$

Alternatively, we can solve for the critical value of the participation cost $\underline{\eta}$ above which a household chooses never to participate in the stock market for given W_0, H_0, and R_0.

$$\Psi((1 - \underline{\eta})W_0, H_0, R_0) = \Psi_0(W_0, H_0, R_0).$$

Because of homogeneity, the solution $\underline{\eta}$ is independent of H_0 and W_0 and is only a function of R_0;

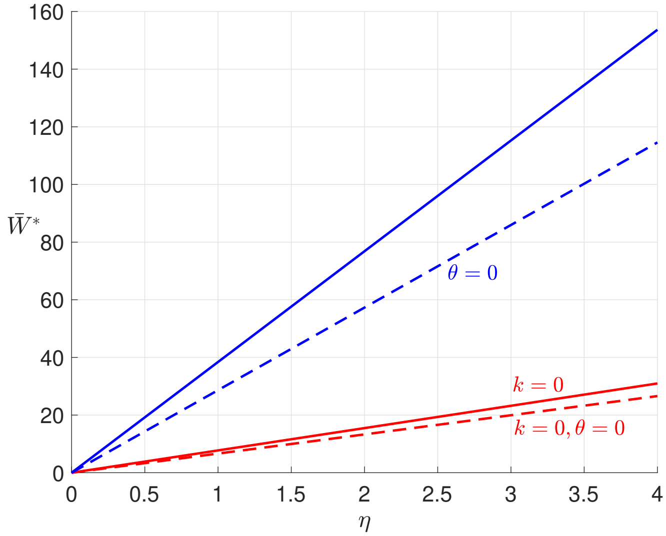


Figure 7: **Wealth threshold $\bar{W}$ under which nonparticipation is optimal against η, both in units of \$1,000.** Default parameters are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$.

we thus denote it as $\underline{\eta}(R_0)$.

In Figure 7, we plot the minimum critical wealth level $\bar{W}^*$ against the participation cost η with and without cointegration when R_0 equals $\bar{R}$. This figure shows that with cointegration, the minimum critical wealth level below which a household will choose never to participate in the stock market is much greater than that without cointegration. For example, if the participation cost is \$1,000, then a household with initial wealth above \$ 6,700 will choose to participate in the stock market when there is no cointegration, but when there is cointegration, even those households that have as much as \$ 38,800 will choose never to participate. If the household does not get utility directly from housing (i.e., $\theta = 0$), because housing is less attractive and the household prefers to invest more in stocks for a given wealth level, the critical wealth level $\bar{W}^*$ is lower, but still as high as \$25,000.

In Figure 8, we plot the critical participation cost $\underline{\eta}$ as a fraction of the initial wealth against the initial conditional expected return of housing $\mu_H(R_0)$ with and without cointegration when R_0 equals $\bar{R}$. This figure shows that with cointegration, the critical participation cost above which a household will choose never to participate in the stock market is much smaller than that without cointegration. For example, when there is no cointegration, the participation cost needs to be as large as 12.92% of the initial wealth to deter a household from stock market participation. In contrast, with cointegration, if $\mu_H(R) = 0.01$, the participation cost only needs to be about 0.65% of the initial wealth to deter a household from ever participating in the stock market. In addition,

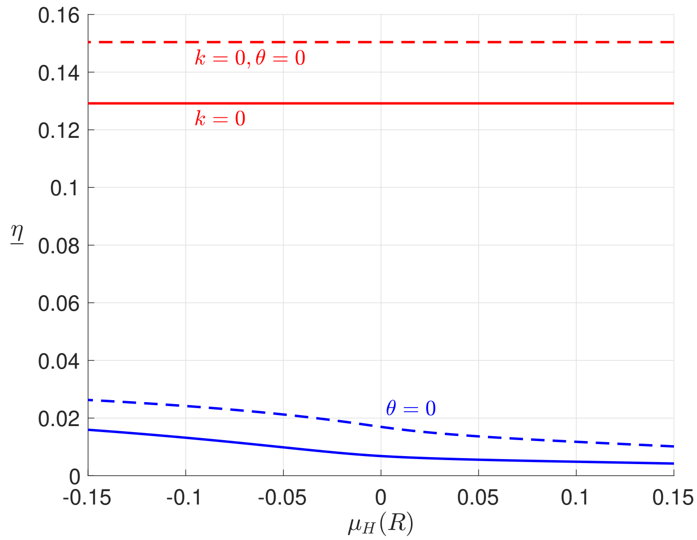


Figure 8: $\underline{\eta}$ **against** $\mu_H(R_0)$**.** Default parameters are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$.

with cointegration, this cost is always below 1.6% for any value of the conditional expected return of housing $\mu_H(R_0)$. These findings suggest that the presence of cointegration can significantly increase the nonparticipation rate in the stock market. If the household does not get utility directly from housing (i.e., $\theta = 0$), because housing becomes less attractive, it is more willing to invest in stocks, and thus requires a higher participation cost for nonparticipation; the critical participation cost as a fraction of the initial wealth $\underline{\eta}$ is higher, but still as low as 1.7% of the initial wealth.

There are many studies aimed at explaining the nonparticipation puzzle. For example, Vissing-Jørgensen (2002) shows that a moderate participation cost can explain half of the nonparticipation observed in data. Cocco (2005) finds that housing crowds out stockholdings, which, together with a sizable stock market participation cost, can explain stock market nonparticipation early in life. Our model complements these extant theories by incorporating the cointegration between the stock and the housing markets, and may strengthen their explanatory power. In particular, our model suggests that the magnitudes of the participation costs in Vissing-Jørgensen (2002) and Cocco (2005) and the degree of ambiguity aversion in Cao, Wang, and Zhang (2005) required to explain nonparticipation would be significantly smaller if cointegration were incorporated.

5 Robustness to the Option of Renting and Illiquidity in the Housing Market

In this section, we examine whether our results are robust to the option of renting and illiqudity in the housing market.

5.1 Option of renting

In our calibrated model with $\theta = 0.3$, housing serves the dual role of an investment vehicle and a consumption good. We have shown that even when the household does not get utility directly from housing (i.e., $\theta = 0$), our main results on non-/limited participation and the highly negative correlation between stock and housing investment remain valid. On the other hand, one might argue that allowing households to rent could significantly weaken the need to buy a house and thus increase stock market participation and investment. In this subsection, we show the robustness of our results in the presence of the renting option.

Assume that the household can rent housing units at a rental rate proportional to the price of the rented property. We denote the rental rate by κ_R. Let h_O and h_R be the fraction of wealth W in houses owned and rented, respectively. Accordingly, the dynamics of wealth W becomes

$$\begin{aligned}\frac{dW_t}{W_t} = &[r - c_t + \zeta_t(\mu_S - r) + h_{Ot}(\mu_H(R_t) - \delta - r) - \kappa_R h_{Rt}]\,dt \\ &+ \sigma_S \zeta_t dB_{St} + \sigma_I h_{Ot} dB_{It} + \sigma_{\varepsilon i} h_{Ot} dB_{it}.\end{aligned} \tag{5.1}$$

The net units of housing at time t are $(h_O + h_R)W/H$ and thus, the household's intertemporal utility becomes

$$\int_0^{\mathcal{T}} e^{-\beta t} \frac{\left({c_t}^{1-\theta}((h_{Ot} + h_{Rt})/H_t)^{\theta} W_t\right)^{1-\gamma}}{1-\gamma} dt.$$

To maximize its expected intertemporal utility, the household chooses the perishable consumption rate c_t and the portfolio weights ζ_t, h_{Ot}, and h_{Rt} of the stock, the housing owned, and the housing rented, respectively. Let $\mathcal{A}_2$ denote the set of all admissible strategies, that is, strategies $(c_t, \zeta_t, h_{Ot}, h_{Rt})$ satisfying standard integrability conditions, the wealth constraint $W_t \geq 0$, the consumption constraints $c_t \geq 0$, the short selling constraints $\zeta_t, h_{Ot}, h_{Ot} + h_{Rt} \geq 0$, and the limited borrowing constraint $\zeta_t + l h_{Ot} \leq 1$. Note that h_{Rt} could be positive or negative. A household rents a house if $h_{Rt} > 0$ and rents out part of its owned house if $h_{Rt} < 0$. The value function is defined as

$$\Psi(W,H,R) := \max_{(c_t,\zeta_t,h_{Ot},h_{Rt})\in\mathcal{A}_2} E\left[\int_0^{\mathcal{T}} e^{-\beta t}\frac{\left({c_t}^{1-\theta}((h_{Ot}+h_{Rt})/H_t)^{\theta}W_t\right)^{1-\gamma}}{1-\gamma}dt\right] \quad (5.2)$$

subject to processes (2.3), (2.7), (5.1), and wealth constraint $W_t \geq 0$. The corresponding HJB equation is given in Appendix A.1.

We also consider a case in which households cannot afford to buy a house (e.g., because a house has a minimum size and the households' wealth is too low) and thus have to rent. In this case, we assume that these households can invest in housing-related securities such as the CSI futures, which have the same price process as the housing index H_{it}.

We use a rental rate of 6.7%, as estimated by Fischer and Stamos (2013), as our default parameter value. Other choices of a rental rate will also be considered. House ownership is affected by the conditional expected return of housing $\mu_H(R)$, as shown in Figure 9. Figure 9 shows that the weight of a house owned increases with the conditional expected return of housing $\mu_H(R)$, while the weight of a house rented decreases and crosses zero with respect to $\mu_H(R)$. Interestingly, the non-/limited stock participation result is strengthened when renting is allowed and a household can buy houses. Recall that when the conditional expected return of housing $\mu_H(R)$ is high, households would like to short stock to finance a purchase of a larger house, but, due to the short-sale constraint, this is impossible. With the rental market opened, households can now buy a larger house and finance part of the purchase with rental revenues from renting out part of the house bought. As a result, compared to the model that does not allow renting, households prefer to buy a larger house and do not participate in the stock market with a greater probability (almost equal to one, as defined in subsection 4.2.2).

When households cannot afford to buy houses but can invest in housing-related securities like CSI futures, Figure 9 shows that the investment in stock is greater and the probability of nonparticipation is lower, compared to the case in which households can afford to buy houses. The rationale behind this is that longing a CSI futures contract does not directly contribute to utility, and thus households invest less in the real estate market. However, our main result that households may choose not to participate in the stock market still holds, because the driving force behind the result is the substitution effect between the stock market and housing market investment, which exists regardless of house ownership.

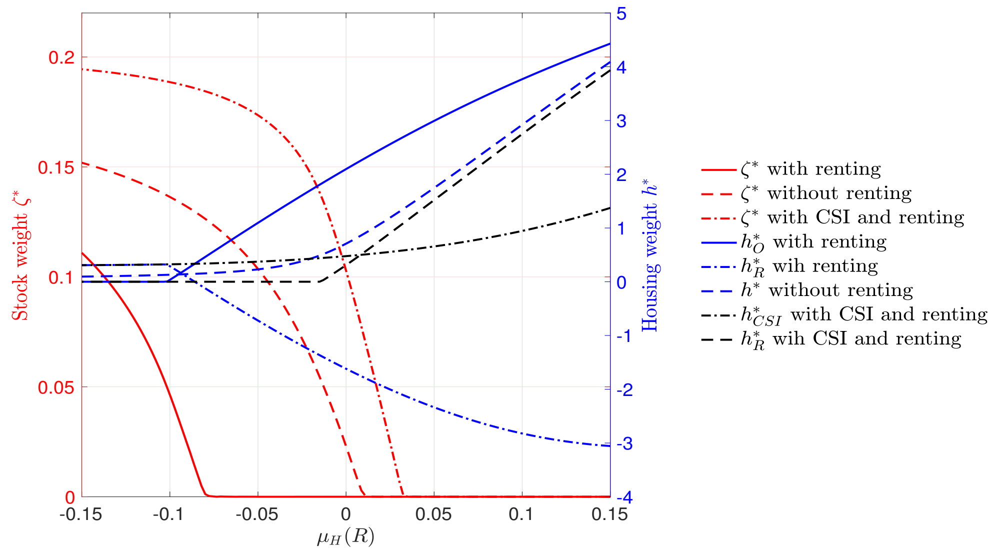


Figure 9: **House ownership and stock investment with renting option.** h_O^*, h_R^*, and ζ^* are the optimal weight of houses owned, houses rented, and stocks when renting is allowed, respectively. h^* and ζ^* are the optimal housing and stock investment when renting is not allowed, as shown in Subsection 4.2. We also consider the case in which households can only rent and invest in CSI futures. The optimal CSI investment is shown as h_{CSI}^* in the plot. Note that with the renting option but without cointegration, the optimal h_O^*, h_R^*, and ζ^* are 1.3588, -1.0677, and 0.1407, respectively. For the model with CSI investment and housing renting but without cointegration, the optimal h_{CSI}^*, h_R^*, and ζ^* are 0.3954, 0.0543, and 0.1407, respectively. Default parameter values: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$, and $\kappa_R = 6.7\%$.

To analyze the average effect of the renting option and the rental rate κ_R on housing and stock investment and on the correlation between the two, we carry out simulations as we did before. The result is reported in Table 4. Panel A shows that when a household can afford to buy a house, as the rental rate increases, stock investment decreases, but both the value of house purchased and the value of house rented out increase. In contrast, Panel B shows that when a household cannot afford to buy a house, as the rental rate increases, stock investment increases, but both the CSI investment and the value of house rented decrease. The difference between the two cases come from the opposite wealth effect from an increase in the rental rate. In the first case, an increase in the rental rate increases the wealth because the household rents out part of the house and thus can get greater revenue, while in the second case, an increase in the rental rate increases the cost of housing consumption. On the other hand, in both cases, the correlations between the stock and the housing/CSI investment are highly negative and tend to increase with the rental rate.

Table 4: **Simulation with renting option.** Default parameter values: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, and $l = 0.2$.

Panel A: With housing and renting

Parameter		Stock Investment (ζ^*)			House Owned (h_O^*)			House Rented (h_R^*)			Investment Correlation		
		Mean	Median	Std	Mean	Median	Std	Mean	Median	Std	Mean	Median	Std
$\kappa_R = 6.7\%$	$k = 0$	0.1407	0.1407	0	1.3588	1.4159	0.1321	−1.0677	−1.1168	0.1076	0	0	0
	$k = 0.1976$	0.0000	0.0000	0.0000	2.9253	3.0930	0.3581	−2.2580	−2.3766	0.2477	−0.7311	−0.8000	0.2608
$\kappa_R = 3.35\%$	$k = 0$	0.1407	0.1407	0	0.8771	0.9087	0.0919	−0.5967	−0.6227	0.0717	0	0	0
	$k = 0.1976$	0.0102	0.0016	0.0202	1.6393	1.6704	0.5437	−0.7996	−0.8900	0.1940	−0.6995	−0.7911	0.2497
$\kappa_R = 1.675\%$	$k = 0$	0.1407	0.1407	0	0.6342	0.6586	0.0744	−0.2782	−0.2978	0.0555	0	0	0
	$k = 0.1976$	0.0109	0.0058	0.0135	1.2921	1.4141	0.2714	−0.7826	−0.8715	0.2502	−0.5629	−0.5589	0.1837

Panel B: With CSI and renting

Parameters		Stock Investment (ζ^*)			CSI Invested (h_{CSI}^*)			House Rented (h_R^*)			Investment Correlation		
		Mean	Median	Std	Mean	Median	Std	Mean	Median	Std	Mean	Median	Std
$\kappa_R = 6.7\%$	$k = 0$	0.1407	0.1407	0	0.3954	0.4137	0.0586	0.0543	0.0543	0.0010	0	0	0
	$k = 0.1976$	0.0755	0.0654	0.0282	0.6075	0.6503	0.1645	0.0988	0.1021	0.0103	−0.9122	−0.9200	0.0560
$\kappa_R = 3.35\%$	$k = 0$	0.1407	0.1407	0	0.3954	0.4137	0.0586	0.1086	0.1086	0.002	0	0	0
	$k = 0.1976$	0.0669	0.0567	0.0287	0.7351	0.7955	0.1904	0.2007	0.2076	0.0219	−0.8726	−0.8473	0.0683
$\kappa_R = 1.675\%$	$k = 0$	0.1407	0.1407	0	0.3954	0.4137	0.0586	0.2172	0.2172	0.0040	0	0	0
	$k = 0.1976$	0.0581	0.0489	0.0257	0.9107	0.9916	0.2190	0.4091	0.4238	0.0469	−0.8803	−0.8623	0.0608

5.2 Illiquidity in the housing market

In our main model, for simplicity of analysis and exposition, we assume there is no transaction cost for buying or selling houses. However, in practice, trading in the housing market can incur significant transaction costs and this may change the effect of the cointegration because of the lower frequency of trading in the housing market. To address this potential concern, we consider the effect of housing market illiquidity in this section.

Following Grossman and Laroque (1990), we assume that to change the house size, a household has to first sell the old house and then purchase a new one of the preferred size, and the household must pay a transaction cost that is proportional to the value of the house sold. More specifically, if the household wants to buy a new house at time τ_i, it is necessary to sell the original house first and pay a transaction cost of $\alpha A_{\tau_i -} H_{\tau_i}$, where $\alpha \in [0, 1)$ represents the proportional transaction cost rate, $A_{t_i -}$ is the size of the original house sold at time t_i-, and H_{t_i} is the market price of the house at that time. After selling, the household buys a new house with size $A_i \geq 0$.[15] Define $\widetilde{W}_t$ as the financial wealth invested in bonds and stocks, π_t as the dollar amount invested in stocks, and $\tilde{C}_t$ as the perishable good consumption. We have

$$
\begin{aligned}
d\widetilde{W}_t &= [r\widetilde{W}_t - \tilde{C}_t + \pi_t(\mu_S - r)]dt + \pi_t \sigma_S dB_{St},\ t \neq \tau_i, && (5.3)\\
dA_t &= -\delta A_t dt,\ t \neq \tau_i, && (5.4)\\
\widetilde{W}_{\tau_i} &= \widetilde{W}_{\tau_i -} + (1-\alpha) A_{\tau_i -} H_{\tau_i} - A_{\tau_i} H_{\tau_i},\ i = 1, 2, ..., && (5.5)\\
A_{\tau_i} &= A_i,\ i = 1, 2, ... && (5.6)
\end{aligned}
$$

The household's objective function in the presence of the illiquid house market is to choose the per-period consumption $\tilde{C}_t$, the stock investment π_t, and the house size A_t to maximize the expected utility, i.e.,

$$
\Psi(\widetilde{W}, A, H, R) = \max_{\tilde{C}_t,\ \pi_t \geq 0,\ (\tau_i, A_i)} \mathbb{E}\left[\int_0^\infty e^{-(\beta+\delta_M)t} \frac{\left(\tilde{C}_t^{1-\theta} A_t^{\theta}\right)^{1-\gamma}}{1-\gamma} dt\right] \quad (5.7)
$$

subject to processes (5.3)–(5.6), (2.7), (2.3), the solvency constraint $\widetilde{W}_t + (1-\alpha)A_t H_t > 0$, the leverage constraint $\pi_t + l(1-\alpha)A_t H_t \leq \widetilde{W}_t + (1-\alpha)A_t H_t$, and the short-sale constraint $\pi_t \geq 0$.

By the homogeneity property, we can make the following transformation:

$$
\Psi(\widetilde{W}, A, H, R) = \frac{1}{1-\gamma}(\widetilde{W} + (1-\alpha)AH)^{1-\gamma} H^{-\theta(1-\gamma)} e^{(1-\gamma)\phi(h,R)}, \quad h = \frac{(1-\alpha)AH}{\widetilde{W} + (1-\alpha)AH},
$$

[15] As in practice, we assume only sellers pay the real estate agent fee.

where h is the ratio of house value to net wealth, and $\phi(.,.)$ is a function to be determined. The corresponding HJB equation and the iterative algorithm for solving it are given in Appendix A.2.

Following Corradin, Fillat, and Vergara-Alert (2014), we set the housing transaction cost to be $\alpha = 10\%$ of the unit's value as a baseline parameter value, including commissions, legal fees, the time cost of searching, and the direct cost of moving possessions. The numerical result is shown in Figure 10. In the presence of transaction costs, there exist an optimal buying ratio $h_B(R)$, an optimal selling ratio $h_S(R)$, and an optimal target ratio of house value to net wealth $h^*(R)$. When the ratio of house value to net wealth is below the optimal buying ratio $h_B(R)$, the household optimally sells the current house and purchases a bigger one such that the new ratio of house value to net wealth jumps upward to the optimal target level $h^*(R)$. When the ratio of house value to net wealth is above the optimal selling ratio $h_S(R)$, the household optimally also sells the current house but purchases a smaller one such that the new ratio of house value to net wealth jumps downward to the optimal target level $h^*(R)$. The area between $h_B(R)$ and $h_S(R)$ is the no-trading region. When the ratio of house value to net wealth falls inside this area, the household does not trade in the housing market.

The function $\phi(h, R)$ satisfies the corresponding HJB equation specified in the Appendix. Value-matching and smooth-pasting conditions hold at the two bounds $h_B(R)$ and $h_S(R)$, and an optimality condition holds at the target point $h^*(R)$. All of these free boundaries depend on the log-ratio R_t.

We plot the optimal ratios of stock value to net wealth $h_B(R)$, $h^*(R)$, and $h_S(R)$ in red-dotted, red-solid, and red-dashed lines, respectively. Figure 10 suggests that the presence of significant illiquidity in the housing market does not change our main result that with cointegration, non-/limited participation in the stock market can be optimal, and the house investment and stock investment are negatively correlated.[16]

To examine the average impact of cointegration, similar to the simulation in Subsection 4.2.1, we set the initial value of the log-ratio R_t to be $R_0 = \bar{R}$. We then simulate 10,000 paths of processes R_t, H_{it}, and $\widetilde{W}_t$ by (2.3), (2.7), and (5.3), respectively. The policy $\{\pi_t, \tilde{C}_t, (\tau_i, A_i)\}$ is chosen from the optimal ones derived by maximizing the objective function. The results presented in Table 5 are averages across 52 states. Table 5 indicates that the presence of significant illiquidity does not change the results: (1) the cointegration effect on average lowers stock investment and increases housing investment and (2) stock investment and house investment are highly negatively correlated (with a mean correlation coefficient of -0.79).

[16] Similar to Grossman and Laroque (1990), when the housing level is at the optimal target, the corresponding stock investment is the lowest. This is because the risk aversion of the household is the highest at the target level, since it takes a significant amount of time before the house size can be changed.

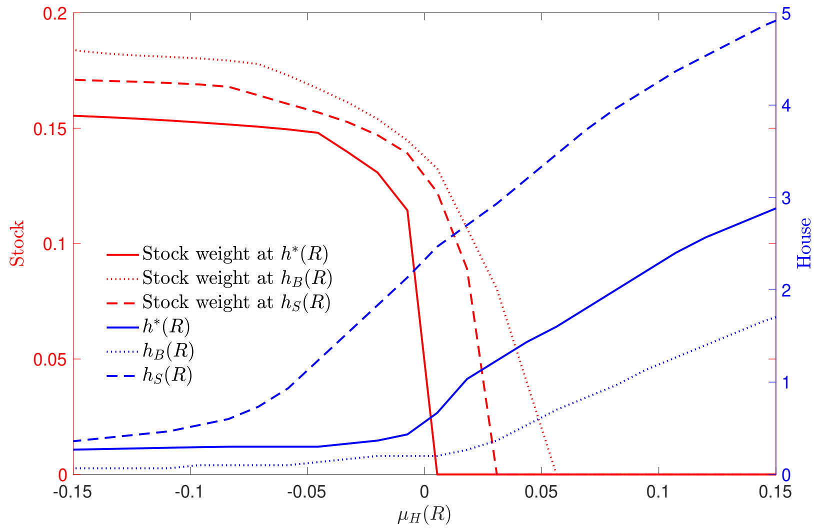


Figure 10: **Optimal stock and housing investment with illiquid housing.** The blue-solid line is the optimal target ratio of house value to net wealth $h^*(R)$. The blue-dashed and blue-dotted lines are the optimal house selling ratio $h_S(R)$ and buying ratio $h_B(R)$, respectively. These three lines are on the right Y-axis. The red-solid, red-dashed, and red-dotted lines are the optimal ratio of stock to net wealth when the ratios of house value to net wealth equal $h^*(R)$, $h_S(R)$, and $h_B(R)$. When there is no cointegration, i.e., $k = 0$, the optimal housing selling ratio $h_S = 0.7333$, the optimal house buy ratio $h_B = 0.26667$, the optimal house target ratio $h^* = 0.4333$, and the optimal stock ratio equals 0.1509, 0.1359, and 0.1332 when the house ratio equals h_B, h_S, and h^*, respectively. Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$, and $\alpha = 0.10$.

Table 5: **Simulation with illiquid housing.** This table reports the simulated average of stock and house investments as percentages of net wealth. Default parameter values are from Table 2: $r = 0.0059$, $\mu_S = 0.0339$, $\sigma_S = 0.1411$, $\lambda = 0.2695$, $\bar{R} = 3.2162$, $\sigma_I = 0.0791$, $\mu_{I0} = 0.0096$, $\sigma_{\varepsilon i} = 0.0316$, $\mu_{\varepsilon i} = 0.0056$, $\beta = 0.0059$, $\delta_M = 0.05$, $\gamma = 10$, $\theta = 0.3$, $\delta = 0$, $l = 0.2$, and $\alpha = 0.10$.

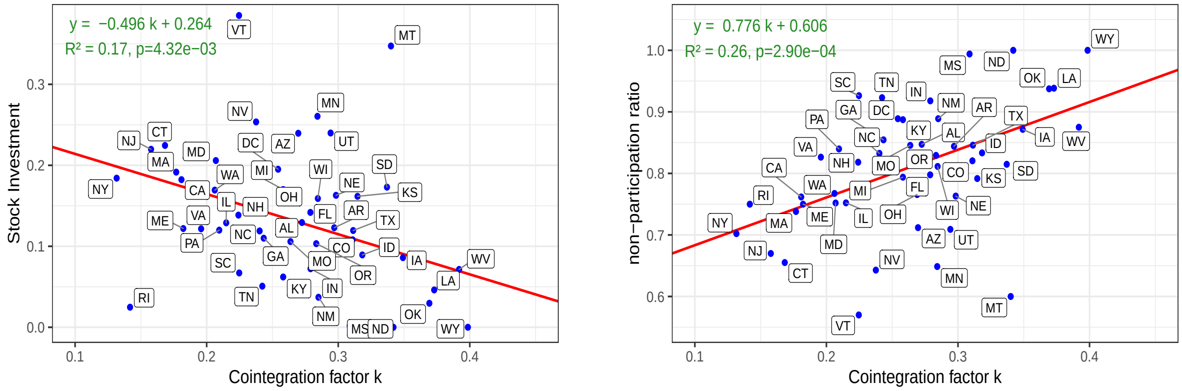
Parameters		Stock Investment			House Investment			Investment correlation		
		Mean	Median	Std	Mean	Median	Std	Mean	Median	Std
$\theta = 0$	$k = 0$	0.1375	0.1346	0.0123	0.2037	0.2018	0.0596	0.1211	0.3986	0.3625
	$k = 0.1976$	0.0273	0.0226	0.0135	0.8264	0.8977	0.2549	-0.6171	-0.6659	0.1851
$\theta = 0.3$	$k = 0$	0.1318	0.1219	0.0097	0.5661	0.5624	0.0466	-0.0518	0.1341	0.2193
	$k = 0.1976$	0.0361	0.0292	0.0260	0.9588	0.9871	0.2378	-0.7860	-0.7251	0.1266

6 Model Prediction and Empirical Evidence

One prediction of our model is that as the degree of the cointegration between housing and stock prices increases, stock investment decreases and stock market nonparticipation increases. To see if this prediction has any empirical support, we next utilize the U.S. cross-state variations of the degree of conintegration, stock investment, and stock market nonparticipation to examine the relations among the three. We use the PSID data of family level in 2015 and 2017 waves, totalling more than 9,000 observations. The value of stockholding is extracted from variable ER65368 in the 2015 wave and ER71445 in the 2017 wave. Financial wealth is calculated as the sum of equity in stocks and the value in safe account, where the value in safe account is the money amount in checking and savings accounts, money market funds, certificates of deposit, government bonds, or treasury bills. In the 2015 wave, the value in safe account is extracted from variable ER61772. In the 2017 wave, the value in safe account is extracted from variable ER67826. We then calculate the average value of equity in stocks, the average ratio of financial wealth invested in stocks,[17] and the proportion of interviewed families that do not invest in stocks for each state. To analyze the cointegration between housing and stock markets in each state, we refer to the monthly Housing Price Index data from 1975 to 2019 from the Federal Housing Finance Agency (FHFA) and estimate the strength of cointegration k for each state. Recall that the larger the value of k, the stronger the cointegration between the housing and stock markets.

Figure 11 shows that, consistent with our model prediction, as the degree of cointegration increases, stock investment decreases and nonparticipation in the stock market increases. For example, Connecticut has a low cointegration between stock and housing prices; as a result, it has high stock investment and low stock market nonparticipation on average.

[17] We can also calculate the equity investment ratio as risky assets divided by the sum of risky assets and safe assets. The risky assets comprise stockholdings, IRAs, and annuity holdings. The safe assets include other assets (net of debt, such as bond and insurance), checking balances, and savings balances, less the principal on the primary residence. The result is similar and not reported here to save space, but available from the authors.



(i) Cointegration effect and stock investment across states (2015 and 2017 waves).

(ii) Cointegration effect and nonparticipation across states (2015 and 2017 waves).

Figure 11: **Cointegration effect on stock investment.** The names of states are abbreviated; for example, “CA" refers to California. The red line is the linear regression result. It is observed that the higher the strength of cointegration between the stock and housing markets, the lower the stock share and the higher the nonparticipation ratio. The p-value reports the significance test result of the coefficient of k in the linear regression.

7 Conclusion

In this paper, we consider the optimal joint choice of stock portfolio and housing of a household when the stock and housing markets are cointegrated. We show that in the presence of cointegration, households significantly reduce stock investment and increase housing investment. As a result, they may choose not to participate in the stock market at all even when there is no participation cost and the unconditional expected return of housing is lower than that of the stock. In the presence of participation cost, the critical wealth level below which households never participate in the stock market is much higher than that in the absence of cointegration, and the critical participation cost level above which households never participate in the stock market is much smaller than that in the absence of cointegration. These results are robust to extensions that incorporate rental alternatives and housing market illiquidity. Our model complements existing studies and can potentially help explain both the puzzle of stock market non-/limited participation and the puzzle of the highly negative correlation between stock and housing investment. We also show empirical evidence that is supportive of some predictions of our model. In particular, across the 50 states of the United States, as the degree of cointegration increases, stock investment decreases and nonparticipation in the stock market increases.

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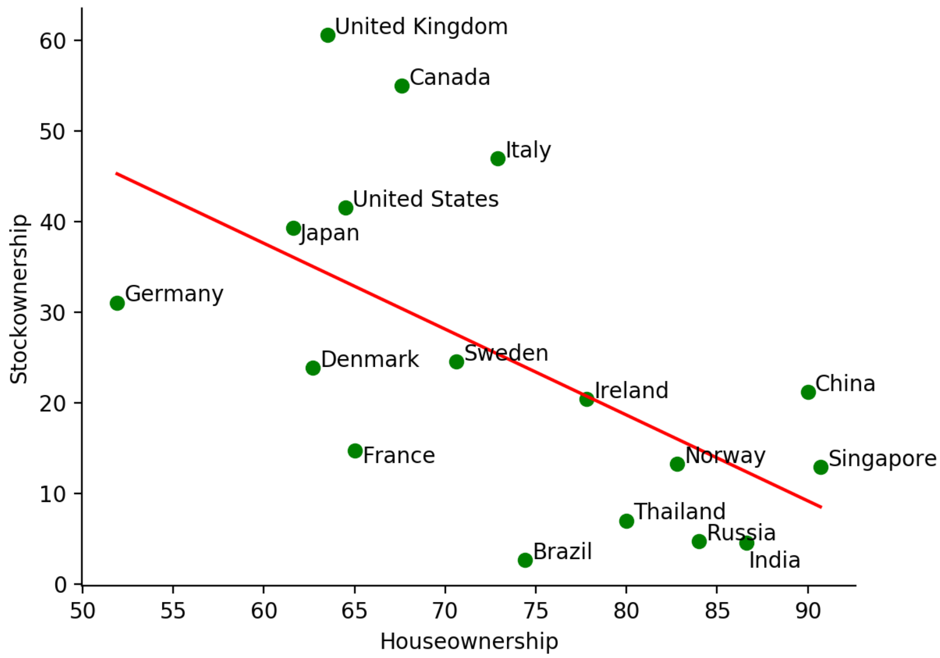

Figure 12: **Homeownership and stock ownership across countries (2015).**

Appendix

In this appendix, we plot some figures to show empirical evidence of the correlations between stock market participation/investment and homeownership/investment across countries and across time. We then provide proofs of the analytical results in the main text.

Empirical Evidence

Figure 12 plots stock ownership (including direct and indirect ownership) against homeownership across 17 countries in 2015, with the red line showing the OLS regression. This figure suggests that significant stock market nonparticipation is an international phenomenon, although standard portfolio choice theories predict close to 100% participation. The highest participation rate is about 61% (United Kingdom) and the lowest about 8% (India). Despite exceptional stock returns, the participation rate in the United States is only about 43%. In addition, Figure 12 suggests that as homeownership increases, stock ownership tends to decrease, with a correlation between the two of about -0.59.

Figure 13 plots stock investment and housing investment across 20 countries in 2015. Figure 13 shows a strong pattern of negative correlation between stock investment and housing investment. Indeed, the correlation between the two is -0.62 on average across these 20 countries.

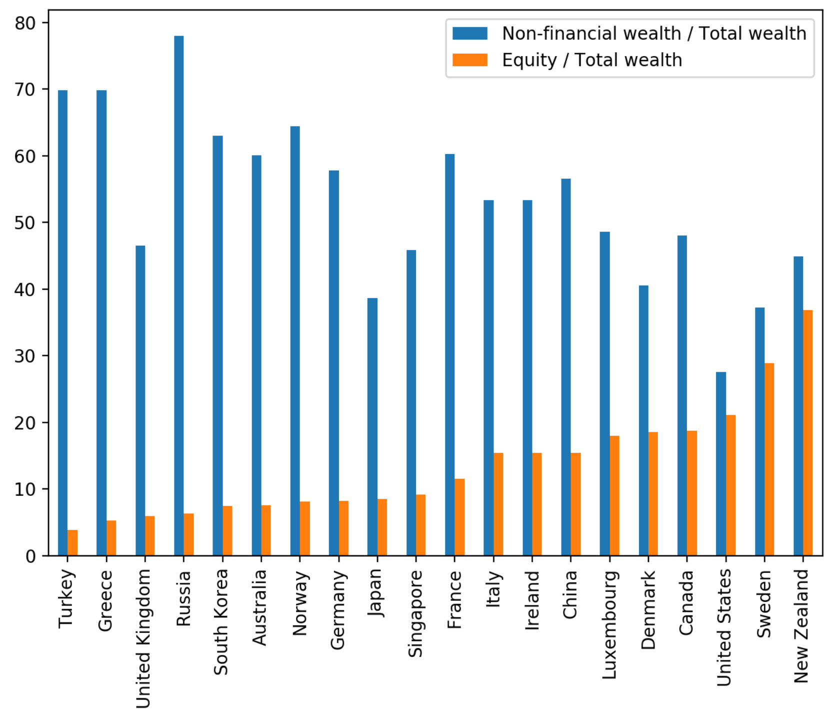

Figure 13: **Nonfinancial wealth and equity across countries (2015)**

Figure 14 plots the correlation between stock investment and housing investment across time over the period from 2000 to 2017 for 16 countries. Figure 14 suggests that stock investment and housing investment are highly negatively correlated across time for most countries.[18] The correlation coefficient is about -0.71 on average. This highly negative correlation is puzzling because it is well known that the contemporaneous correlation between stock and housing prices is low (around 0.07) and standard optimal investment theories imply a low negative correlation of around zero (see the noncointegration cases in Tables 3, 4, and 5).

[18]The only exceptions are Australia, Canada, and Singapore, where the correlations are positive. The positive correlations found in these countries may be related to special immigration and real estate policies. For example, for Australia and Canada, immigration is encouraged and immigrants, who invest in both housing and stocks, tend to be wealthy.

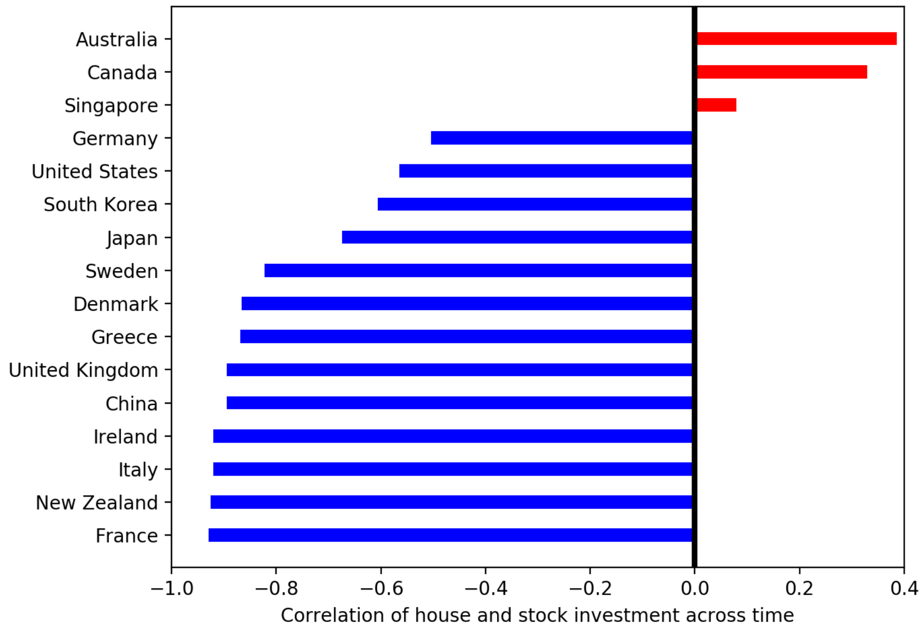

Figure 14: **Housing and stock investment correlations across time (2000-2017)**

A.1 HJB Equation for the Model with Option of Renting

The associated HJB equation for the household's optimization problem (5.2) is

$$\begin{aligned}\max_{c,\,\zeta,\,h_O\geq 0;\,\zeta+lh_O\leq 1;\,h_R}&\Big\{(\frac{1}{2}\sigma_S^2\zeta^2+\frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)h_O^2)W^2\Psi_{WW}+\frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)H^2\Psi_{HH}\\&+\frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)\Psi_{RR}+(\sigma_I^2h_O-\lambda\sigma_S^2\zeta)W\Psi_{WR}+(\sigma_I^2+\sigma_{\varepsilon i}^2)Hh_OW\Psi_{WH}+\sigma_I^2H\Psi_{HR}\\&+[r-c+(\mu_S-r)\zeta+(\mu_H(R)-\delta-r+\kappa_R)h_O-\kappa_R(h_O+h_R)]W\Psi_W\\&+\mu_H(R)H\Psi_H+k(\bar{R}-R)\Psi_R-(\beta+\delta_M)\Psi+\frac{\left(c^{1-\theta}((h_O+h_R)/H)^\theta W\right)^{1-\gamma}}{1-\gamma}\Big\}=0\end{aligned} \quad \text{(A.1-1)}$$

for $W>0$, $H>0$, $R\in\mathbb{R}$.

Using the homogeneity property of the value function, we can reduce the dimensionality of the problem by the following transformation:

$$\Psi(W,H,R)=\frac{1}{1-\gamma}W^{1-\gamma}H^{-\theta(1-\gamma)}e^{(1-\gamma)u(R)}$$

for some function u. Then equation (A.1-1) can be reduced to

$$\max_{c,\,\zeta,\,h_O\geq 0;\,\zeta+lh_O\leq 1;\,h_R}\Big\{\frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)[u''+(1-\gamma)u'^2]+\Big[(\sigma_I^2h_O-\lambda\sigma_S^2\zeta-\theta\sigma_I^2)(1-\gamma)+k\bar{R}-kR\Big]u'$$
$$-\frac{1}{2}\gamma(\sigma_S^2\zeta^2+(\sigma_I^2+\sigma_{\varepsilon i})h_O^2)+\frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i})h_O^2\theta(\theta(1-\gamma)+1)-(\sigma_I^2+\sigma_{\varepsilon i})\theta(1-\gamma)h_O+r-c$$
$$+(\mu_S-r)\zeta+(\mu_H(R)-\delta-r+\kappa_R)h_O-\kappa_R(h_O+h_R)-\theta\mu_H(R)-\frac{\beta+\delta_M}{1-\gamma}$$
$$+\frac{1}{1-\gamma}c^{(1-\theta)(1-\gamma)}(h_O+h_R)^{\theta(1-\gamma)}e^{-(1-\gamma)u}\Big\}=0,\quad R\in\mathbb{R}.$$

A.2 HJB Equation for the Model with an Illiquid Housing Market

The value function as given in (5.7) satisfies the following HJB equation:

$$\max\Big\{\max_{\tilde{C},\,0\leq\pi\leq\widetilde{W}+(1-l)(1-\alpha)AH}\Big(\frac{1}{2}\sigma_S^2\pi^2\Psi_{\widetilde{W}\widetilde{W}}+\frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)H^2\Psi_{HH}+\frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)\Psi_{RR}$$
$$-\lambda\sigma_S^2\pi\Psi_{\widetilde{W}R}+\sigma_I^2H\Psi_{HR}+[r\widetilde{W}-\tilde{C}+(\mu_S-r)\pi]\Psi_{\widetilde{W}}+\mu_H(R)H\Psi_H$$
$$+k(\bar{R}-R)\Psi_R-\delta A\Psi_A-(\beta+\delta_M)\Psi+\frac{\Big(\tilde{C}^{1-\theta}A^\theta\Big)^{1-\gamma}}{1-\gamma}\Big),\qquad\text{(A.2-1)}$$
$$\max_{0<Q\leq\frac{\widetilde{W}+(1-\alpha)AH}{l(1-\alpha)H+\alpha H}}\Psi(\widetilde{W}+(1-\alpha)AH-QH,Q,H,R)-\Psi(\widetilde{W},A,H,R)\Big\}=0$$

in $\Omega=\{(\widetilde{W},A,H,R):\widetilde{W}+(1-\alpha)AH>0,A>0,H>0,R\in\mathbb{R}\}$.

By the homogeneity property, we can do the following transformation:

$$\Psi(\widetilde{W},A,H,R)=\frac{1}{1-\gamma}(\widetilde{W}+(1-\alpha)AH)^{1-\gamma}H^{-\theta(1-\gamma)}e^{(1-\gamma)\phi(h,R)},\quad h=\frac{(1-\alpha)AH}{\widetilde{W}+(1-\alpha)AH},$$

where h is the ratio of house value to net wealth. The new function $\phi(h,R)$ satisfies the following HJB equation:

$$\max\Big\{\mathcal{L}\phi,\quad\max_{0\leq h'\leq 1/l}\Big(\phi(h',R)+\log\frac{1-\alpha}{1-\alpha+\alpha h'}\Big)-\phi(h,R)\Big\}=0,\qquad\text{(A.2-2)}$$

where

$$\mathcal{L}\phi = \max_{\zeta\in[0,1-lh]} \Big\{ \frac{1}{2}[\sigma_S^2\zeta^2 + (\sigma_I^2+\sigma_{\varepsilon i}^2)(1-h)^2]h^2[\phi_{hh} + (1-\gamma)\phi_h^2] + \frac{1}{2}(\lambda^2\sigma_S^2+\sigma_I^2)[\phi_{RR}+(1-\gamma)\phi_R^2]$$
$$+ [\lambda\sigma_S^2\zeta + \sigma_I^2(1-h)]h[\phi_{hR} + (1-\gamma)\phi_h\phi_R] + \big[\gamma\sigma_S^2\zeta^2 - (\sigma_I^2+\sigma_{\varepsilon i}^2)(\theta(1-\gamma)+\gamma h)(1-h)$$
$$- (\mu_S - r)\zeta + (\mu_H(R) - r - \delta)(1-h)\big]h\phi_h + \big[(1-\gamma)\sigma_I^2(h-\theta) - \lambda(1-\gamma)\sigma_S^2\zeta$$
$$+ k(\bar{R}-R)\big]\phi_R - \frac{1}{2}\gamma\sigma_S^2\zeta^2 + \frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)\theta(\theta(1-\gamma)+1) - \frac{1}{2}(\sigma_I^2+\sigma_{\varepsilon i}^2)(2\theta(1-\gamma)+\gamma h)h$$
$$+ r(1-h) + (\mu_S - r)\zeta + \mu_H(R)(h-\theta) - \delta h - \frac{\beta+\delta_M}{1-\gamma} - \eta(1-h\phi_h)^{-\frac{p}{1-p}} h^{\frac{\theta(1-\gamma)}{1-p}} e^{-\frac{1-\gamma}{1-p}\phi} \Big\}$$

with $\zeta = \pi/(\widetilde{W} + (1-\alpha)AH)$ and the optimal consumption satisfies

$$\tilde{c}^* = \frac{\tilde{C}^*}{\widetilde{W} + (1-\alpha)AH} = (1-\theta)^{\frac{1}{1-p}}(1-h\phi_h)^{-\frac{1}{1-p}} h^{\frac{\theta(1-\gamma)}{1-p}} e^{-\frac{1-\gamma}{1-p}\phi}.$$

We can solve (A.2-2) numerically. Define $M(R) := \max_{0\le h\le 1/l} \left\{\phi(h,R) + \log\left(\frac{1-\alpha}{1-\alpha+\alpha h}\right)\right\}$. The algorithm is given as follows:

(1) Set an initial guess of $M_0(R)$;

(2) Given $M_i(R)$, use the penalty method with finite difference scheme[19] to solve

$$\max\Big\{\mathcal{L}\phi, \quad M_i(R) - \phi\Big\} = 0, h \in (0, 1/l);$$

(3) Let $M_{i+1}(R) := \max_{0\le h'\le 1/l} \left\{\phi(h',R) + \log\left(\frac{1-\alpha}{1-\alpha+\alpha h'}\right)\right\}$;

(4) If $||M_{i+1} - M_i|| <$ tolerance then stop; otherwise go to Step 2.

A.2.1 Verification theorem

In this subsection, we provide the verification theorem that captures households' optimal investment policy. We focus on the general illiquid model in Subsection 5.2, while the liquid benchmark model in Section 2 is a degenerated case.

Theorem 2. ***(Verification Theorem)*** *Let $\Psi(\widetilde{W}, A, H, R)$ be a smooth solution to the HJB equation (A.2-1) and satisfy the transverality condition, i.e.,*

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-(\beta+\delta_M)t}\Psi(\widetilde{W}_t, A_t, H_t, R_t)\right] = 0,\ \forall \widetilde{W}_t + (1-\alpha)A_tH_t > 0, A_t > 0, H_t > 0, R_t \in \mathbb{R}.$$

[19] For the penalty method, see, e.g., Dai and Zhong (2010).

In addition, let

$$\tilde{C}_t^*(\widetilde{W}_t, A_t, H_t, R_t) = \left(\frac{\Psi_W A^{-\theta(1-\gamma)}}{1-\theta}\right)^{\frac{1}{(1-\theta)(1-\gamma)-1}},$$

$$\pi_t^*(\widetilde{W}_t, A_t, H_t, R_t) = \underset{0 \le \pi_t \le \widetilde{W}_t + (1-l)(1-\alpha)A_t H_t}{\arg\max} \left\{\frac{1}{2}\sigma_S^2 \pi^2 \Psi_{\widetilde{W}\widetilde{W}} - \lambda \sigma_S^2 \pi \Psi_{\widetilde{W}R} + (\mu_S - r)\pi \Psi_{\widetilde{W}} - \delta A \Psi_A - \eta A^{\frac{-\theta(1-\gamma)}{(1-\theta)(1-\gamma)}} \Psi_W^{\frac{(1-\theta)(1-\gamma)}{(1-\theta)(1-\gamma)-1}}\right\},$$

where $\eta = (1-\theta)^{\frac{1}{1-(1-\theta)(1-\gamma)}} - \frac{(1-\theta)^{\frac{(1-\theta)(1-\gamma)}{1-(1-\theta)(1-\gamma)}}}{1-\gamma}$*, all the partial derivatives are evaluated at* $(\widetilde{W}_t, A_t, H_t, R_t)$*, and*

$$Q_t^*(\widetilde{W}_{t-}, A_{t-}, H_t, R_t) = \underset{0 < Q \le \frac{\widetilde{W}_{t-} + (1-\alpha)A_{t-}H_t}{\alpha H_t + l(1-\alpha)H_t}}{\arg\max} \Psi(\widetilde{W}_{t-} + (1-\alpha)A_{t-}H_t - Q_t H_t, Q_t, H_t, R_t),$$

$$\tau_i^* = \inf\{t > \tau_{i-1}^* : \ \Psi(\widetilde{W}_{t-}, A_{t-}, H_t, R_t) = \Psi(\widetilde{W}_{t-} + (1-\alpha)A_{t-}H_t - Q_t^* H_t, Q_t^*, H_t, R_t)\},$$

$$A_i^* = Q_{\tau_i^*}^*(\widetilde{W}_{\tau_i^*-}, A_{\tau_i^*-}, H_{\tau_i^*}, R_{\tau_i^*}).$$

Then, $\Psi(\widetilde{W}, A, H, R)$ *coincides with the value function as given in (5.7), and* $\Theta^* = \{\tilde{C}_t^*, \pi_t^*, (\tau_i^*, A_i^*)\}$ *is the corresponding optimal investment policy.*

Proof. Given any admissible investment policy $\Theta = \{\tilde{C}_t, \pi_t, (\tau_i, A_i)\}$, we denote by $(\widetilde{W}_t, A_t, H_t, R_t)$ the stochastic processes generated by policy Θ for notional convenience. Define $\mathcal{O}_n := \{(\widetilde{W}, A, H, R) : \frac{1}{n} \le \widetilde{W} + (1-\alpha)AH \le n, \frac{1}{n} \le A \le n, \frac{1}{n} \le H \le n, |R| \le n\}$ and a sequence of stoppting times $T_n := n \wedge \inf\{t \ge 0 : (\widetilde{W}_t, A_t, H_t, R_t) \notin \mathcal{O}_n\}$.

By the generalized Ito's formula,

$$
\begin{aligned}
&e^{-(\beta+\delta_M)T_n}\Psi(\widetilde{W}_{T_n}, A_{T_n}, H_{T_n}, R_{T_n}) \\
=\ &\Psi(\widetilde{W}, A, H, R) - \int_0^{T_n} e^{-(\beta+\delta_M)u}\frac{\left(\tilde{C}_u^{1-\theta}A_u^{\theta}\right)^{1-\gamma}}{1-\gamma}du \\
&+ \int_0^{T_n} e^{-(\beta+\delta_M)u}\mathcal{L}\Psi(\widetilde{W}_u, A_u, H_u, R_u)du \\
&+ \int_0^{T_n} e^{-(\beta+\delta_M)u}\big(\pi_u\sigma_S\Psi_{\widetilde{W}} - \lambda\sigma_S\Psi_R\big)(\widetilde{W}_u, A_u, H_u, R_u)dB_{Su} \\
&+ \int_0^{T_n} e^{-(\beta+\delta_M)u}\big(\sigma_I H_t\Psi_H + \sigma_I\Psi_R\big)(\widetilde{W}_u, A_u, H_u, R_u)dB_{Iu} \\
&+ \int_0^{T_n} e^{-(\beta+\delta_M)u}\sigma_{\varepsilon i}H_t\Psi_H(\widetilde{W}_u, A_u, H_u, R_u)dB_{iu} \\
&+ \sum_{\tau_i<T_n} e^{-(\beta+\delta_M)\tau_i}\big(\Psi(\widetilde{W}_{\tau_i}, A_{\tau_i}, H_{\tau_i}, R_{\tau_i}) - \Psi(\widetilde{W}_{\tau_i-}, A_{\tau_i-}, H_{\tau_i}, R_{\tau_i})\big),
\end{aligned}
\tag{A.2-3}
$$

where

$$
\begin{aligned}
\mathcal{L}\Psi =\ &\frac{1}{2}\sigma_S^2\pi^2\Psi_{\widetilde{W}\widetilde{W}} + \frac{1}{2}(\sigma_I^2 + \sigma_{\varepsilon i}^2)H^2\Psi_{HH} + \frac{1}{2}(\lambda^2\sigma_S^2 + \sigma_I^2)\Psi_{RR} - \lambda\sigma_S^2\pi\Psi_{\widetilde{W}R} \\
&+ \sigma_I^2 H\Psi_{HR} + [r\widetilde{W} - \tilde{C} + (\mu_S - r)\pi]\Psi_{\widetilde{W}} + \mu_H(R)H\Psi_H \\
&+ k(\bar{R} - R)\Psi_R - \delta A\Psi_A - (\beta+\delta_M)\Psi + \frac{\left(\tilde{C}^{1-\theta}A^{\theta}\right)^{1-\gamma}}{1-\gamma}.
\end{aligned}
$$

Since Ψ is the solution to HJB equation (A.2-1), the second integral term and the last term in the right-hand side of (A.2-3) are nonpositive. Bellman's principle of optimality suggests that these two terms equal zero under the optimal policy $\Theta^* = \{\tilde{C}_t^*, \pi_t^*, (\tau_i^*, A_i^*)\}$. The three Ito integrals under expectation equal zero because $\Psi_{\widetilde{W}}, \Psi_H, \Psi_R$ are bounded when $(\widetilde{W}_u, A_u, H_u, R_u)$ is in the bounded domain $\mathcal{O}_n$ during $[0, T_n]$. Taking expectation in (A.2-3), we have

$$
\Psi(\widetilde{W}, A, H, R) \geq E\left[\int_0^{T_n} e^{-(\beta+\delta_M)u}\frac{\left(\tilde{C}_u^{1-\theta}A_u^{\theta}\right)^{1-\gamma}}{1-\gamma}du\right] + E\left[e^{-(\beta+\delta_M)T_n}\Psi(\widetilde{W}_{T_n}, A_{T_n}, H_{T_n}, R_{T_n})\right].
$$

As analyzed above, the equality above holds only for the claimed optimal strategy $\Theta^* = \{\tilde{C}_t^*, \pi_t^*, (\tau_i^*, A_i^*)\}$. As $n \to \infty$, T_n tends to infinity with probability 1. By the transversality condition of Ψ and the dominant convergence theorem, the first expectation above converges to the original utility function $E\left[\int_0^{\infty} e^{-(\beta+\delta_M)u}\frac{\left(\tilde{c}_t^{1-\theta}A_t^{\theta}\right)^{1-\gamma}}{1-\gamma}dt\right]$ and the second expectation goes to zero. Equality holds for the claimed optimal policy Θ^* and Ψ coincides with the original objective function. This completes the proof. $\square$

A.3 Conditional Expected Return of Housing and Stock

The sources of the stock market data and the house price series are Standard & Poor's and the Case Shiller Home Price Indicies, respectively, on December 1 from 1890 to 2017; both are in inflation-adjusted Novemeber 2019 dollars. As shown in Section 3, the residual process $R_t = \log I_t - \lambda \log S_t$ with $\lambda = 0.2695$. To compare the conditional expected return of housing and stock, we analyze the growth rates of house price and stock price when R_t falls below its long-term limit $\bar{R}$. We divide the whole sample periods into two groups by the sign of R_t. The result is shown in Figure 15. The upper panel shows the residual process $R_t = \log I_t - 0.2695 \log S_t$. The middle panel depicts the house growth rate $\log(I_{t+1}/I_t)$ at the state of $R_t < \bar{R}$. The lower panel depicts the stock growth rate $\log(S_{t+1}/S_t)$ at the state of $R_t < \bar{R}$. The average house growth rate when $R_t < \bar{R}$ from observation year 1890 (1953) to 2017 equals 0.0126 (0.0139), while the average stock growth rate when $R_t < \bar{R}$ from observation year 1890 (1953) to 2017 equals -0.0028 (0.0132). It shows that the house price grows faster than the stock price in the state of $R_t < \bar{R}$, supporting the notion that when the conditional expected return of housing is high relative to that of stock, households prefer to stay away from the stock market, as analyzed in the main body of our paper.

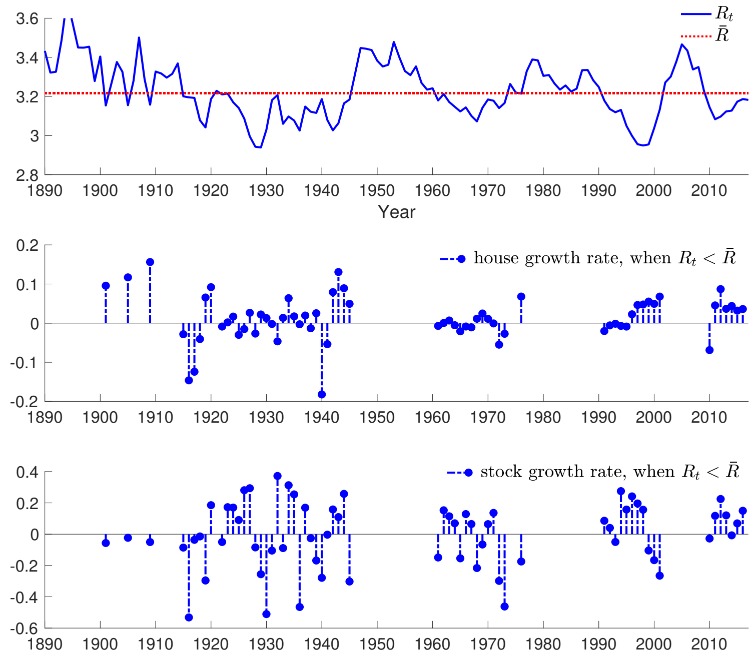

Figure 15: **Growth rate of house price and stock price.** The upper panel shows the residual process R_t. The middle panel depicts the house index growth rate $\log(I_{t+1}/I_t)$ at the state of $R_t < \bar{R}$. The lower panel depicts the stock growth rate $\log(S_{t+1}/S_t)$ at the state of $R_t < \bar{R}$. The average house growth rate when $R_t < \bar{R}$ from 1890 (1953) to 2017 equals 0.0126 (0.0139), while the average stock growth rate when $R_t < \bar{R}$ from 1890 (1953) to 2017 equals -0.0028 (0.0132). This implies that the house price grows faster than the stock price when the ratio of house price to stock price is low.