Investing for the Long Run

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Abstract

This paper studies long term investing by an investor that maximizes either expected utility from terminal wealth or from consumption. We introduce the concepts of a generalized stochastic discount factor (SDF) and of the minimum price to attain target payouts. The paper finds that the dynamics of the SDF needs to be captured and not the entire market dynamics, which simplifies significantly practical implementations of optimal portfolio strategies. We pay particular attention to the case where the SDF is equal to the inverse of the growth-optimal portfolio in the given market. Then, optimal wealth evolution is closely linked to the growth optimal portfolio. In particular, our concepts allow us to reconcile utility optimization with the practitioner approach of growth investing. We illustrate empirically that our new framework leads to improved lifetime consumption-portfolio choice and asset allocation strategies.

Keywords

stochastic discount factor, minimum pricing, optimal portfolio, growth optimal portfolio

JEL Classification

G11, G13

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1 Introduction

Long term investing is an important concern of individual investors (e.g. saving for retirement) and of financial intermediaries (e.g. managing pension funds). A popular advise for a moderate-risk US retirement portfolio is the 60/40 rule of thumb, i.e. to hold 60% of marketable wealth in stocks and the remainder (40%) in bonds (Campbell and Viceira (2002)). However, it remains unclear when such a two-fund portfolio strategy is optimal, why these proportions are (roughly) optimal and what exactly constitutes the risky asset position. This paper aims to address such questions. In particular, we discuss practical aspects of optimal long-term investing with particular focus on identifying settings where dynamic asset allocation strategies remain rather simple.

There exists an extensive literature on the optimization of expected utility from terminal wealth and consumption-savings portfolios, including, e.g. Merton (1971), Epstein and Zin (1989), Duffie and Epstein (1992b), Campbell and Viceira (1999), Chacko and Viceira (2005), and Kraft et al. (2013). In this literature, the martingale technique has turned out to be a promising approach; see Cox and Huang (1989), Cvitanić and Zapatero (2004) and Pennacchi (2008) for details. Our paper is rooted in our observation that the martingale technique expresses desired asset allocations in terms of, so-called, stochastic discount factors (henceforth SDFs, also known as pricing kernels).

This paper draws attention to properties of SDFs and relaxes the restrictive assumptions of classical no-arbitrage theory, which relies on the risk neutral pricing paradigm. Instead, we argue for the additional and rather natural requirement that the SDF be tradeable. We introduce and discuss the notion of minimum pricing, which allows us to use a generalized martingale technique and thereby show that our notion of tradeable SDF plays an important role in asset allocation. This addresses two of our initial questions: when are two-fund strategies optimal and what characterizes the risky asset position?

We stress the fact that the, so-called, growth optimal portfolio of the given investment universe (henceforth GP), see Kelly (1956) and Merton (1971), plays a central role in the solutions of most optimal investment strategies. Along these lines, minimum pricing relates to the benchmark pricing theory of Platen (2002), and Platen and Heath (2010), which opens a much wider, and thus more realistic, modeling world than available under the classical no-arbitrage paradigm.

Our paper points out that the inverse of the GP is the natural choice of SDF and so we focus on this SDF thereafter. We draw attention to the stochastic dynamics of the GP and show that the optimal investment strategy simplifies dramatically when the dynamics of the GP is Markovian. In this, for a modeler appealing, yet also rather realistic situation, the resulting optimal strategies allocate assets into a few funds only. In particular, we show for a Markovian GP model with one driving source of uncertainty, that the optimal asset allocation consists of two funds: the riskless asset and the GP.

Finally, we carry out an empirical evaluation of investment strategies that are derived from a particular model. Although the model does not satisfy classical no-arbitrage assumptions, its long-term dynamics are realistic and permit asset allocation in a now practically tractable manner. Our long-term investment strategies relate to higher long-term growth and achieve objectives less expensively than possible under the classical paradigm. Along these lines we also address the initial quantitative question of what constitutes an optimal proportion between riskless and risky assets.

The literature has made tremendous advances in the theoretical understanding of asset management. It appears that many mathematical aspects have been clarified to a large extent. However, there remain intriguing problems that impede their practical relevance beyond simple situations. We contribute to this literature by addressing three formidable difficulties.

First, long term investing addresses changes in the investment opportunity set over time (Campbell and Viceira (2002)) but solving the associated dynamic optimization problem is intrinsically hard. Moreover, the known solutions suggest that the optimal dynamic trading strategies are, in general, rather complex. Our approach, however, reduces the asset allocation to investing into a few funds, including a single risky fund that is closely related to the stochastic dynamics of the tradeable SDF.

Second, the current literature provides the impression that qualitative insights depend crucially on the underlying setup, in particular, on the concrete parametric formulation of the investment objective. For example, different preference specifications, e.g. additive utility versus recursive utility preferences, seem to lead to asset allocations that are widely different. The current paper provides a unifying theory that clarifies in optimal asset allocation the crucial link to tradeable SDFs and thereby stresses commonalities. We show that our results hold, whenever investors that maximize utility from terminal wealth or consumption aims for the least expensive strategy.

MacLean et al. (2010) and Davis and Lleo (2015) note that several legendary investors, including John Maynard Keynes, Warren Buffet and Bill Gross follow strategies that maximize long-term growth. Moreover, Christensen (2012) provides an excellent description of an older lively literature that recommends using such strategies. He explains forcefully the differences between growth maximization and utility maximization. However, the current paper employs the concept of minimum pricing to shows that utility maximization in the presence of a tradeable (unknown) SDF is intrinsically linked to growth maximization. Thereby, we provide a fresh look at the older growth optimal investing literature and additional support for common long-term practitioner strategies.

Third, the practical implementation of theoretical investment strategies is currently a daunting task. Typically, it involves dynamic asset allocation in a wide range of securities to capture the impact of the theoretically possible driving factors, which represents a seemingly infeasible challenge in implementation. To make matters worse, modeling the entire market dynamics and reliably estimating, in particular, means but also covariances of a large number of assets is hard in practice; see DeMiguel et al. (2009). Our approach provides a considerable simplification in this endeavor as it calls attention to modeling the dynamics of the GP. In particular, we introduce a realistic, univariate long-term dynamics for the GP and show that it can be reliably implemented leading to superior, two fund investment strategies.

The paper is organized as follows: Section 2 introduces our setup, while the following Section discusses structural properties of price dynamics. Section 4 uses these to characterize optimal portfolio choice in terms of the GP. This draws our attention to modeling the price dynamics of the GP in Section 5 which allows us to gain structural insights into portfolio construction. The following Section links our results to well-known ones from the literature. Section 7 evaluates the performance of our model in comparison to other approaches. Section 8 concludes the paper.

2 The Market Environment

We assume a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, where P denotes the real-world probability measure. $\mathcal{A}_0$ is assumed to be the trivial σ-algebra. The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{0 \leq t < \infty}$ describes the evolution of information in the market, i.e. at any time $t \in [0, \infty)$ the σ-algebra $\mathcal{A}_t$ describes the information available at that time.

2.1 Assets and Portfolios

We consider a market composed of $d+1$ primary security accounts. These securities include the locally risk-free base line asset $S_t^0 = 1, t \in [0, \infty)$. In addition, the market is composed of d risky primary security accounts $S^j = (S_t^j)_{t \in [0,\infty)}, j = 1, \ldots, d$, where all payments (e.g. dividends, income or interest) are reinvested, and S_t^j is denominated in units of S_t^0. Note that one could also include, e.g. human capital, intellectual property, production facilities, or other (non-)financial assets in the given investment universe, as long as they can be traded.

In line with Merton (1971), we assume perfect capital markets: trading in the securities takes place continuously in time; there are no transaction costs; at any point in time locally risk-free, instantaneous investing and borrowing is possible; the securities are infinitely divisible, and short sales with full use of proceeds are allowed.

For simplicity, we illustrate our approach for continuous dynamics, only, but dynamics involving jumps can be handled similarly. Also, for illustration purposes, we denominate our securities in units of the locally risk-free savings account, our baseline asset. However, our methodology allows many denominations. From a consumption-savings portfolio perspective, for example, one may prefer to denominate prices in units of the consumer-price index (CPI), i.e. employing an inflation rate accruing account as denominator.

The dynamics of the risky j-th primary security account S_t^j is given through the stochastic differential equation (SDE)

$$dS_t^j \quad = \quad S_t^j \left(a_t^j dt + \sum_{k=1}^{d} b_t^{j,k} dW_t^k \right) = S_t^j \left(a_t^j dt + b_t^{j\top} dW_t \right), \tag{1}$$

$t \in [0, \infty)$. Here, $S_0^j > 0$ is given for $j = 1, \ldots, d$, and $W^1, W^2, \ldots, W^d$ are d independent Brownian motions modeling the traded uncertainty. We denote by $\top$ the vector/matrix transpose and by $W_t = (W_t^1, \ldots, W_t^d)^\top$ the d-dimensional vector of the values of the independent Brownian motions at time t. We define $a_t^S = (a_t^1, \ldots, a_t^d)^\top$ as the d-dimensional (instantaneous) appreciation rate vector, and $b_t^S = (b_t^{j,k})_{j,k=1,\ldots,d} = (b_t^{j\top})_{j=1,\ldots,d}$ as the $d \times d$-dimensional (instantaneous) volatility matrix. Both stochastic processes are assumed to be adapted to the filtration $(\mathcal{A}_t)_{0 \le t < \infty}$ and may be driven also by sources of uncertainty other than the traded uncertainty modeled by W. This allows one to capture state variables that describe some (potential) incompleteness in this market. We assume that the respective system of SDEs has a unique strong solution, see e.g. Section 7.7 in Platen and Heath (2010).

Throughout this paper we adopt the following assumption:

Assumption 1 *Almost surely under the real-world probability measure P, the volatility matrix b_t^S is invertible with inverse $b_t^{S,-1}$ for all $t \in [0, \infty)$, and the process $(b_t^S)_{t \in [0,\infty)}$ satisfies the condition $\int_0^T \sum_{j,k=1}^d (b_t^{j,k})^2 dt < \infty$ for all $T \in [0, \infty)$.*

The invertibility assumption for the volatility matrix ensures that the d-dimensional vector

$$\theta_t = b_t^{S,-1} a_t^S \tag{2}$$

is well-defined at all times, and we refer to θ_t as the market-price-of-risk vector at time t. A consequence is that θ_t becomes the volatility of the unique and finite growth optimal portfolio (GP), as we discuss later. The second condition in Assumption 1 ensures sufficient integrability of the stochastic Itô integrals involved.

2.2 Preferences

We introduce an indicator variable $\chi \in \{0,1\}$, fix $T \in [0,\infty)$ and denote by V_0 the investor's initial wealth, as well as, by $\pi = (\pi_t)_{0\leq t\leq T}$ the (vector) process of wealth weights, with $\pi_t = (\pi_t^1, \pi_t^2, \ldots, \pi_t^d)^\top$. Here, π_t^j denotes the relative weight of the time $t \in [0,T]$ investment in the risky primary security S_t^j, $j = 1,\ldots,d$. Note that $\pi_t^0 = 1 - \sum_{j=1}^d \pi_t^j$ is the fraction invested in the locally riskfree baseline asset. (Negative weights indicate borrowing.) We denote by $C = (C_t)_{0\leq t\leq T}$ the agent's consumption process, financed by the initial capital V_0 and through the investment strategy π. Then, the agent's wealth is given by the (self-financing) process $(V_t^\pi)_{0\leq t\leq T}$, satisfying by (2) the SDE

$$
\begin{aligned}
dV_t^\pi &= \left(\pi_t^\top a_t^S V_t^\pi - \chi C_t\right) dt + (\pi_t^\top b_t^S) V_t^\pi dW_t && (3)\\
&= \left(\pi_t^\top b_t^S \theta_t V_t^\pi - \chi C_t\right) dt + (\pi_t^\top b_t^S) V_t^\pi dW_t. && (4)
\end{aligned}
$$

For a given parameter $\varepsilon \geq 0$ and given indicator variable $\chi \in \{0,1\}$, our objective is to determine a process $J = (J_t)_{t\in[0,T]}$ that solves the optimization

$$
J_t = \max_{\pi,C} E\left[\int_t^T \chi f(C_s, J_s, s) ds + \varepsilon B(V_T^\pi) \middle| \mathcal{A}_t\right], \tag{5}
$$

for all $t \in [0,T)$. Here, at any time $t \in [0,T)$, the maximization is taken over the remaining time period $[t,T]$ by choosing a respective portfolio weight process π and consumption process C with associated self-financing wealth process (4). Furthermore, we employ in (5) the conditional expectation $E[\cdot|\mathcal{A}_t]$ under the real-world probability measure P, given the information at time t, encapsulated in $\mathcal{A}_t$. The time t information includes the value of the portfolio V_t^π at time t. The process J characterizes the, so-called, life-time utility derived from consumption.

Throughout this paper we adopt, for simplicity, the following additional assumption on the utility functions involved:

Assumption 2 *For given $s \in [0,T]$ and $l \in \mathbb{R}$ the functions $f(\cdot,l,s), B(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ are twice differentiable, strictly increasing, strictly concave and fulfill the Inada conditions, i.e.* $\lim_{x\downarrow 0} f'(x,l,s) = \lim_{x\downarrow 0} B'(x) = \infty$ *and* $\lim_{c\to\infty} f'(c,l,s) = \lim_{x\to\infty} B'(x) = 0$.

The parameters ε and χ allow us to study a variety of portfolio choice problems. The case $\chi = 0$, $\varepsilon > 0$ corresponds to the problem of a price taking agent who maximizes expected utility derived from terminal wealth V_T^π, as in Merton (1971), i.e. a portfolio choice problem without consumption (a so-called asset allocation problem); the function B plays here the role of the utility function.

The case $\chi = 1$ corresponds to a consumption-savings problem (or intertemporal consumption and portfolio choice problem) either with bequest ($\varepsilon > 0$) or without bequest ($\varepsilon = 0$). The parameter ε controls the importance of bequest relative to the utility derived from consumption. Our specification in (5) captures, among others, the usual case of time-separable utility, as well as, the continuous-time version of, so-called, Epstein-Zin preferences introduced within the context of stochastic differential utility by Duffie and Epstein (1992a), see Appendix A. We will discuss in Subsections 6.2 and 6.3 examples of time-additive preferences over consumption and of preferences over terminal that are covered. Whenever needed, we assume existence and uniqueness of stochastic differential utility in the sense of Duffie and Epstein (1992a).

In a general market setting, the conditional expectation (5) is usually a rather complex function of many state variables. This makes practical applications of the theoretical optimal strategy difficult, if not seemingly impossible: (too) many quantities have to be accurately modeled and estimated to obtain a practically useful strategy. It is well known that the estimation of parameters characterizing expected returns requires longer observation windows than available in reality, see e.g. Kan and Zhou (2007) and DeMiguel et al. (2009). Furthermore, estimated covariance matrices of returns are stochastic themselves and, in general, extremely difficult both to estimate and to invert with meaningful outcomes, especially for a large number of assets, see e.g. Bai and Ng (2002), Ludvigson and Ng (2007) and Okhrin and Schmid (2006). Notwithstanding these difficulties in general, the main emphasis of this paper is to point out realistic and practically relevant situations, where the conditional expectation in (5) becomes a function of the current time t and potentially a few, well observable state variables.

3 Structural Properties of Price Dynamics

A crucial determinant of asset allocation is the price dynamics of securities. To prepare for our asset allocation analysis in the next section we, therefore, characterize some of their structural properties. The first subsection discusses pricing through a (generalized) stochastic discount factor (SDF). The second subsection stresses the tradeability of the SDF. The following subsection identifies tradeable SDFs, i.e. they are given as the inverse of the, so-called, growth optimal portfolio (GP). The fourth subsection discusses the benchmark pricing theory and introduces minimum pricing. The fifth subsection explains how minimum pricing is useful for analyzing optimal portfolio allocations.

3.1 Stochastic Discount Factor

The absence of arbitrage opportunities in market dynamics is a common assumption in classical financial economics, often employed to ensure well-defined financial markets. The workhorse underpinning modeling efforts is the Fundamental Theorem of Asset Pricing, which asserts that the absence of, so-called, classical arbitrage opportunities is equivalent to the existence of an equivalent risk-neutral probability measure; see Harrison and Kreps (1979) and Delbaen and Schachermayer (2006). In the language of financial economics this is equivalent to the existence of a stochastic discount factor (SDF) with respective properties; see e.g. Cochrane (2001). Unfortunately, staying in the classical no-arbitrage framework comes at the cost of restrictive and, potentially, unrealistic modeling assumptions that impede its application in practice; see e.g. Loewenstein and Willard (2000) and Baldeaux et al. (2015).

This paper takes a more general approach, in line with a growing literature that avoids classical no-arbitrage assumptions and recommends working in a wider modeling world than the one that assumes the existence of an equivalent risk-neutral probability measure; see e.g. Loewenstein and Willard (2000), Platen (2002), Platen (2006), Karatzas and Kardaras (2007), Fernholz and Karatzas (2010) and Platen and Heath (2010). Compared to the more general approach we pursue, the consequences of the classical, more restrictive assumptions for short term investing may be minor. However, the long-term consequences are substantial, as we demonstrate.

By generalizing concepts and ideas presented in the wide literature on classical asset pricing that have been well summarized in Cochrane (2001), let us introduce the following assumption:

Assumption 3 *There exists an adapted, strictly positive process $(F_s)_{0\leq s\leq T}, T\in[0,\infty)$ such that $0<F_t<\infty$ almost surely for $t\in[0,T]$, and such that for all primary securities S_t^j, $j=0,\dots,d$, and value processes of self-financing portfolios the SDE for the product of F_t and the value or price process is driftless, in particular, $(F_t\cdot S_t^j)_{0\leq t\leq T}$ is driftless. To be even more precise, we assume that we can write*

$$dF_t = F_t(a_t^F dt + b_t^{F\top} dW_t + c_t^{F\top} d\bar{W}_t) \tag{6}$$

with $F_0=1$ for a suitable adapted real valued process a^F, a suitable adapted vector process $b^F=(b^{F1},\dots,b^{Fd})^\top$, and a suitable adapted vector process $c^F=(c^{F1},\dots,c^{Fn})^\top$, with non-

negative integer n. Here, the non-traded uncertainty $\bar{W} = (\bar{W}^1, \ldots, \bar{W}^n)^\top$ is given by a vector process of n independent Brownian motions that are independent of the Brownian motion vector process W, the traded uncertainty. We refer to F with these properties as a stochastic discount factor (SDF)*.*

This assumption is weaker than the classical one, where the product process $(F_t \cdot S_t^j)_{0 \leq t \leq T}$ must form an $(\underline{\mathcal{A}}, P)$-martingale. Here we only require these processes to form local martingales. Assumption 3 allows us to discuss in a more flexible, transparent and unified manner different trade-offs in terms of pricing and asset allocation that a modeler faces. In particular, the well-established literature on asset pricing relates conveniently, as a special case, to our approach.

The common modeling approach starts with assuming the existence of a risk-neutral probability measure. Such a restriction blends together two conditions that we both relax in Assumption 3. First, under classical no-arbitrage assumptions, one typically assumes that for any price process $(A_t)_{t \in [0,T]}$ the product $(A_t F_t)_{0 \leq t \leq T}$ forms a true martingale, which leads to the pricing rule

$$A_t F_t = E[A_s F_s | \mathcal{A}_t] \tag{7}$$

for all $0 \leq t \leq s \leq T$. In Assumption 3 we only request that $A_t F_t$ is driftless, and, thus, forms a local martingale. Second, the pricing rule (7) recovers so-called classical risk-neutral pricing, see Cochrane (2001), *if and only if* $(F_t)_{0 \leq t \leq T}$ forms a true martingale. The risk-neutral probability measure is in this case characterized via the Radon-Nikodym density process $(F_t)_{0 \leq t \leq T}$. We underline the crucial fact that Assumption 3 permits a much wider class of models where F_t needs only to form an $(\underline{\mathcal{A}}, P)$-local martingale, and the pricing rule (7) is only one of many possible ones.

As we will see later on, it is a rather strong assumption of the common classical modeling approach that an asset price multiplied by an SDF needs to form a true martingale. We emphasize once more that we do *not* assume this property in the current paper. Consequently, the pricing rule (7) does not need always to hold in our more general setting. Furthermore, even if the pricing rule would hold, this may not necessarily imply that $(F_t)_{0 \leq t \leq T}$ itself needs to form a true martingale. This does not mean that we could not have in our market price processes that result from formally applying the risk-neutral pricing rule. However, we only request in Assumption 3 that the resulting price processes, when multiplied with the SDF, have no drift, thus form local martingales.

While the local martingale property in Assumption 3 is significantly relaxed (compared to the martingale property), it imposes still restrictions. These allow us to derive the following conclusions: First, Assumption 3 in (6) implies with respect to the locally risk-free asset

$S_t^0 = 1$ that $a_t^F = 0$. Furthermore, by equations (1) and (2), it follows via the Itô formula with respect to all risky securities S_t^j with $j = 1, \dots, d$ that

$$\begin{aligned} d(F_t S_t^j) &= F_t a_t^j S_t^j dt + F_t b_t^{j\top} S_t^j dW_t + S_t^j F_t b_t^{F\top} dW_t + S_t^j F_t c_t^{F\top} d\bar{W}_t + b_t^{j\top} b_t^F F_t S_t^j dt \\ &= S_t^j F_t \left(a_t^j + b_t^{j\top} b_t^F \right) dt + S_t^j F_t \left(b_t^{j\top} + b_t^{F\top} \right) dW_t + S_t^j F_t c_t^{F\top} d\bar{W}_t \\ &= S_t^j F_t \left\{ b_t^{j\top} \left(\theta_t + b_t^F \right) dt + \left(b_t^{j\top} + b_t^{F\top} \right) dW_t + c_t^{F\top} d\bar{W}_t \right\}. \end{aligned}$$

For this SDE to be driftless, we must have $b_t^{j\top} \left(\theta_t + b_t^F \right) = 0$ for all $j = 1, \dots, d$. This means that we need to have $b_t^F = -\theta_t$, i.e. by (6) we obtain

$$\frac{dF_t}{F_t} = -\theta_t^\top dW_t + c_t^{F\top} d\bar{W}_t. \tag{8}$$

We emphasize that this is the generic form of an SDF in our market.

3.2 Tradeability of SDF

Our definition of an SDF in Assumption 3 explicitly allows the SDF to be driven by traded and potentially also by non-traded uncertainty, see equation (6). Therein, the SDF $F = (F_t)_{0 \le t \le T}$, satisfying equation (8) can be represented as

$$F_t = F_0 \exp \left\{ -\frac{1}{2} \int_0^t \theta_s^\top \theta_s ds - \int_0^t \theta_s^\top dW_s \right\} \tilde{F}_t, \tag{9}$$

with factor

$$\tilde{F}_t = \exp \left\{ -\frac{1}{2} \int_0^t c_s^{F\top} c_s^F ds + \int_0^t c_s^{F\top} d\bar{W}_s \right\}. \tag{10}$$

The process $\tilde{F}$ is driven by non-traded uncertainty and, therefore, cannot be hedged through dynamic asset allocation. The decomposition (9) shows that introducing non-tradeable uncertainty into the SDF is equivalent to introducing a non-hedgeable factor process $\tilde{F}$ in the construction of the SDF. Theoretically, this factor process seems to provide wider generality for the notion of an SDF. However, as we will argue now, it appears not to provide any usable flexibility in practical valuations or hedging.

The factor $\tilde{F}$ is not unique but, by definition, any choice of $\tilde{F}$ must not be driven by any traded uncertainty. Taking a formal look at this fact from the valuation perspective, one notes that, in order to make the pricing rule (7) work, say, under classical assumptions, the process $\tilde{F}$ has to be a true martingale. As a consequence of basic rules of stochastic calculus, the non-traded uncertainty $\tilde{F}$ in (9) does not impact in (7) the prices of any hedgeable payoffs and is, therefore, redundant in such a valuation approach. As such it is superfluous when pricing replicable claims.

There is an additional conceptual argument against introducing a superfluous factor $\tilde{F}$ into an SDF, taken from the perspective of practical implementation of optimal portfolio choices. We will see in the next sections that optimal portfolios and their price processes become driven by the SDF. Therefore, when the factor $\tilde{F}$ is not replicable from traded security prices, i.e. when non-traded sources of uncertainty drive the SDF, then optimal portfolios and their price processes become driven by non-tradeable uncertainty, i.e. require investing in non-tradeable factors. To implement such strategies, we would have to make non-tradeable uncertainty tradeable, which defeats the purpose.

In summary, the generality of the forms of the dynamics of the SDF satisfying (8), which involves the factor $\tilde{F}$, is artificial and evaporates from the perspective of practical feasibility. Based on these insights, in the remainder of this paper we remove from the SDF, given in (9), the superfluous non-tradeable factor $\tilde{F}$. Instead we offer more practically relevant flexibility in the current paper than under classical assumptions: we allow $(A_t F_t)_{t\in[0,T]}$ to form strict local martingales (our approach) instead of true martingales (classical approach). We will demonstrate that this flexibility is important for making long-term investing less expensive.

Consequently, throughout this paper, we work with an SDF satisfying (8) with $c_t^F = 0$, that is the SDF satisfies from now on the SDE

$$dF_t = -F_t \theta_t^\top dW_t, \tag{11}$$

for $t \geq 0$ and $F_0 = 1$.

3.3 The Inverse Growth Optimal Portfolio as SDF

The previous subsection introduced the notion of a tradeable SDF, which we identify in this subsection and for which we discuss here implications for pricing.

The growth optimal portfolio (GP) with value $V_t^{\pi^{GP}} = V_t^{GP}$ at all times $t \in [0,T]$, characterized as the portfolio with $V_0^{GP} = 1$, is unique and has wealth weights

$$\pi_t^{GP} = b_t^{S,-1,\top} \theta_t \tag{12}$$

at time $t \in [0,T]$ (without consumption, $C = 0$). Based on equation (4) it fulfills the SDE

$$\begin{aligned} dV_t^{GP} &= (\pi_t^{GP\top} a_t^S) V_t^{GP} dt + \pi_t^{GP\top} b_t^S V_t^{GP} dW_t, \\ &= V_t^{GP} \left\{ (\theta_t^\top \theta_t) dt + \theta_t^\top dW_t \right\} \end{aligned} \tag{13}$$

for $t \geq 0$ with $V_0^{GP} = 1$. By Assumption 1 the GP exists in our market, i.e. due to its finite expected instantaneous growth rate $\frac{1}{2}\theta_t^\top \theta_t$, it is strictly positive and finite at any finite time. By application of the Itô formula to $1/V_t^{GP}$ we find that

$$F_t = \frac{1}{V_t^{GP}} \tag{14}$$

for $t \geq 0$ solves the SDE (11). Thus, the inverse of the GP is a suitable choice of an SDF in the sense of Assumption 3.

Additionally, the SDF (14) is tradeable. By Assumption 3, any other candidate for a tradeable portfolio that can be used as a numéraire should, when used as denominator for the GP, generate a strictly positive local martingale, which by Fatou's Lemma is then a supermartingale, see Platen and Heath (2010) Theorem 10.2.1. But also the GP, when used as denominator for the candidate portfolio, is a local martingale, and, thus, a supermartingale. By Jensen's inequality, these two supermartingales can only equal the constant one. Therefore, the tradeable SDF (14) is *unique*.

In the remainder of this paper we set F according to equation (14), or equivalently according to the SDE (11).

3.4 Benchmark Pricing Theory

The benchmark pricing theory, proposed by Platen (2002), assumes only the existence of the GP, which it calls benchmark, and makes it its central building block. Any value multiplied with F_t is called a benchmarked value. In particular, we denote by $\hat{S}_t^j = \frac{S_t^j}{V_t^{GP}}$ the respective benchmarked value of the primary risky security S_t^j, $j = 1 \ldots, n$, and consider the *benchmarked wealth* process $\hat{V}^\pi = \frac{V_t}{V_t^{GP}}$ without consumption ($C = 0$). (The next subsection will study *benchmarked (consumption-adjusted) wealth* processes $\hat{G}^\pi$.)

Through the existence of the GP, see Platen and Heath (2010), there is no economically meaningful arbitrage in the market, in the sense that no strategy can generate in finite time a strictly positive portfolio with infinite wealth from finite initial capital. Under classical no-arbitrage assumptions, Long (1990) made the observation that the risk-neutral price can be recovered by choosing the GP as numéraire and the real world probability measure as pricing measure. This means, the pricing rule (7) with $F_t = 1/V_t^{GP}$ recovers in the classical setting the risk-neutral price for a replicable contingent claim. Therefore, the GP is also termed the numéraire portfolio (NP).

The benchmark pricing theory goes much further than reformulating classical risk-neutral pricing, it drops the classical assumptions, see Platen and Heath (2010), and requests instead only the existence of the GP. It employs then the GP as denominator, numéraire and benchmark. In summary, the benchmark pricing theory obtains a unique and *tradeable* SDF by setting F_t equal to the locally riskless baseline security, denominated in units of the GP. This means that the SDF F_t represents the inverse of the discounted GP. Therefore, by setting $\tilde{F}_t = 1, t \geq 0$ in (10), we identify via (14) the inverse F_t^{-1} with the discounted benchmark V_t^{GP}.

In the general setting of the benchmark pricing theory, several self-financing portfolios can replicate the same payoff. While this means that the classical Law of One Price may not hold any longer in the more general modeling world we consider, this does not permit creation of any economically meaningful arbitrage: the absence of economically meaningful arbitrage should be interpreted in the sense that there does not exist a strictly positive portfolio that has an infinite instantaneous growth rate such that it could become infinite in finite time; see Platen and Heath (2010). Most importantly, from a conceptual viewpoint we note that the failure of the Law of One Price allows several pricing rules to coexist. For instance, one can formally apply the risk-neutral pricing rule even when F does not form a true martingale. However, as we will see now, there exists always a minimum price, which is uniquely determined.

As indicated earlier, the growth optimal portfolio (GP) maximizes expected logarithmic utility derived from terminal wealth. The GP is the best performing, strictly positive portfolio in the sense that in the long run its value surpasses almost surely the value of all other strictly positive portfolios, see e.g. Platen and Heath (2010), Theorem 10.5.1. This also means that the inverse of the GP, our SDF, is the inverse of the portfolio that leads, in the long run, almost surely to lower values than the inverse of any other strictly positive portfolio could achieve. When we employ the inverse of the GP as SDF we, therefore, can intuitively expect it to lead us to the lowest prices possible among all potential price processes for a targeted payoff. This means, when one employs the benchmarked savings account as SDF, one obtains intuitively via the pricing rule (7) the minimal possible price process.

To make this precise we introduce the notion of a fair price or value process:

Definition 4 *A price or value process* $(A_t)_{0 \leq t \leq T}$ *is called fair, when it satisfies the pricing rule* (7) *with the inverse of the GP as SDF* $F = 1/V^{GP}$*, that is*

$$\frac{A_t}{V_t^{GP}} = E\left[\left. \frac{A_s}{V_s^{GP}} \right| \mathcal{A}_t \right] \tag{15}$$

for $0 \leq t \leq s \leq T$.

This means that the benchmarked value $\hat{A}_t = \frac{A_t}{V_t^{GP}} = F_t A_t$ forms an $(\underline{\mathcal{A}}, P)$-martingale.

In formula (15) pricing is performed under the real-world or objective probability measure P with the inverse of the GP as SDF. In Platen and Heath (2010) this is referred to as *real-world pricing* and the formula (15) is called the real world pricing formula.

Assumption 3 requires that all benchmarked prices form local martingales. This implies that all benchmarked wealth processes (without consumption) are local martingales and, thus, supermartingales. It is known, see Lemma A1 in Du and Platen (2016), that a martingale is the minimal non-negative supermartingale that delivers at a bounded stopping time a targeted nonnegative integrable payout. This fact implies that the *benchmarked* wealth process, which aims for the minimal possible initial expense to deliver some specified payoff, must be a martingale, and, thus, by Definition 4 it must be a fair price process. The respective martingale exists and is unique because its benchmarked value at a given time is the conditional expectation of the benchmarked portfolio value at maturity under the available information.

We summarize this insight as follows:

Corollary 5 *The minimal price or value process that delivers a targeted payoff needs to be fair. More expensive price or value processes are possible that when benchmarked form non-negative local martingales, and, thus, supermartingales.*

It is important to emphasize that our assumptions assure the existence of the GP, and avoid the restrictive classical no-arbitrage assumptions. In particular, we avoid the request on the existence of an equivalent risk-neutral probability measure. This opens up a much wider modeling world than provided under classical no-arbitrage assumptions.

3.5 Minimum Pricing of Consumption-Savings Investments

The previous subsection introduced the notion of real-world pricing and showed that it leads to minimal possible prices. Let us now apply this pricing approach for consumption-savings investments.

Recall that for any traded security with value S_t^j we denote by $\hat{S}_t^j = \frac{S_t^j}{V_t^{GP}} = S_t^j F_t$ its benchmarked value. For a given consumption-savings investment strategy (π, C) we denote for $t \in [0, T]$ the benchmarked wealth process by $\hat{V}_t^\pi = V_t^\pi F_t$, and the consumption-adjusted wealth process G^π by

$$G_t^\pi = V_t^\pi + \chi \int_0^t C_s ds, \tag{16}$$

leading with

$$\hat{G}_t^\pi = G_t^\pi F_t, \tag{17}$$

to the *benchmarked (consumption-adjusted) wealth* process $\hat{G}^\pi$. Equations (4) and (8) imply for the benchmarked (consumption-adjusted) value $\hat{G}_t^\pi$ the SDE

$$\begin{aligned} d\hat{G}_t^\pi &= F_t(dV_t^\pi + \chi C_t dt) - G_t^\pi F_t \theta_t^\top dW_t - (\pi_t^\top b_t^S) V_t^\pi F_t \theta_t dt \\ &= (\pi_t^\top b_t^S \hat{V}_t^\pi - \hat{G}_t^\pi \theta_t^\top) dW_t. \end{aligned} \tag{18}$$

This means that the benchmarked self-financing portfolio process is driftless and, therefore, forms a local martingale. In addition, $\hat{G}_t^\pi$ is nonnegative. Thus, again by Fatou's Lemma, $\hat{G}^\pi$ is a supermartingale.

Note that benchmarked (consumption-adjusted) wealth processes may exist in our setting that are strict supermartingales but not martingales. This means that these portfolios may not be fair. However, it is interesting to further study those *particular* portfolios which *are* fair.

Due to the classical Law of One Price, usually, the literature does not emphasize that the investor cares about the minimal possible initial expense. Nevertheless, in our more general setting, where several self-financing portfolios may exist that deliver the same payout, this objective now appears as a reasonable component of the optimization. Lower initial capital needed for a payout translates into extra capital that allows for higher consumption. Therefore, we aim for consumption-savings profiles and associated wealth processes that require minimal initial capital.

In the remainder we consider the optimization problems where the investor restricts herself to fair consumption-savings portfolios in the sense of Definition 4, such that her objective is to determine a process $J = (J_t)_{t \in [0,T]}$ that solves, as in (5), the optimization

$$J_t = \max_{(\pi,C)} E\left[\int_t^T \chi f(C_s, J_s, s) ds + \varepsilon B(V_T^\pi) \middle| \mathcal{A}_t \right], \tag{19}$$

for all $t \in [0, T)$, now subject to *fair* and self-financing wealth dynamics (4).

The utility optimization problem (19) may be different from our initial utility optimization (5), since cost minimization is not an explicit goal in (5) and so the investor may seem to have opportunities to trade off its benefits against potential utility gains. Whenever needed, we assume existence and uniqueness of stochastic differential utility in the sense of Duffie and Epstein (1992a) over this restricted set of strategies.

Along the way, significant practical benefits become available. As we will see in the next sections, the restriction to fair strategies permits us to proceed *analogous* to the well-known martingale technique. Intuitively, the optimization problem (19) means that the investor carries out a two-step optimization along the lines of the martingale technique: First, she looks for a fair optimal consumption profile for any given initial cost and then, in the second step, ensures that the initial cost matches her initial wealth.

Under classical no-arbitrage assumptions, the pricing rule (7) is the risk-neutral pricing rule and has been the prevailing one. As we will see later on, by applying formally risk-neutral pricing to our more general setting (which is possible) one obtains a benchmarked portfolio that is a local martingale. However, it may not be a martingale. In any case, it would be a supermartingale and, when not a martingale, would be more expensive than the respective martingale, the fair price. Real-world pricing according to (15) is minimum pricing and yields fair consumption-adjusted wealth processes. The restrictive assumption on the existence of an equivalent risk-neutral probability measure is, in our general setting, no longer enforced and portfolios can be constructed with less expensive dynamic asset allocation strategies than those obtained from formally applied risk-neutral pricing.

Minimum pricing (real-world pricing) is to be distinguished from risk-neutral pricing, where a risk-neutral probability measure Q is assumed to exist in a probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, Q)$ s.t. savings account discounted price processes become martingales under the assumed risk-neutral probability measure. Under the classical no-arbitrage paradigm it is well-known that an SDF can be used to define the risk-neutral probability measure, see Cochrane (2001). Risk-neutral and real-world pricing are equivalent under the existence of an equivalent risk-neutral probability measure, see equation (10.4.5) in Platen and Heath (2010). However, they differ in theory and in actual markets, see e.g. Baldeaux et al. (2015).

By formally applying risk-neutral pricing, which is currently mostly done in practice, one simply ignores the possibility that the benchmarked savings account may be, in reality, a strict local martingale, and, thus, a strict supermartingale. In such a case, the formally obtained risk-neutral price would be usually more expensive than the minimum price which is obtained via the real-world pricing formula (15).

An important question that arises in our optimization is the tradeability of the fair optimal consumption-savings portfolio. The next sections will show that the fair portfolio is tradeable once the SDF is tradeable. In summary, *fair* portfolio strategies encapsulate already an important first optimization step that does not arise under classical assumptions, where one formally applies risk-neutral pricing.

4 Optimal Benchmarked Portfolio Choice

The previous sections discuss the central role of the GP as benchmark or numéraire in pricing. This section assumes real-world pricing to obtain minimum prices and uses the properties derived to re-express the optimal wealth process (as well as the optimal consumption process) in terms of the GP and the baseline security.

Let us consider a budget-feasible, fair consumption-savings and terminal wealth plan and postpone to a further discussion the issue, whether such a plan would be tradeable. The problem then reduces to finding the optimal, budget-feasible plan, i.e. we look for processes C and J, and a random variable V_T^π that maximize the right-hand side in equation (19).

Real-world pricing tells us that the current benchmarked value of a consumption-savings investment strategy (π, C) with benchmarked consumption-adjusted value process $\hat{G}^\pi$ is given by the conditional expectation

$$\hat{G}_t^\pi \;=\; E\left[\hat{G}_T^\pi \middle| \mathcal{A}_t\right] = E\left[\int_0^T \chi \frac{C_s}{V_s^{GP}} ds + \frac{V_T^\pi}{V_T^{GP}} \middle| \mathcal{A}_t\right].$$

Therefore, we have for the benchmarked portfolio value $\hat{V}_t^\pi = \frac{V_t^\pi}{V_t^{GP}}$ and the benchmarked consumption $\hat{C}_t = \frac{C_t}{V_t^{GP}}$ the equality

$$\hat{V}_t^\pi \;=\; E\left[\int_t^T \chi \hat{C}_s ds + \hat{V}_T^\pi \middle| \mathcal{A}_t\right], \tag{20}$$

$t \in [0, T]$. Using a Lagrange multiplier λ to capture the initial budget constraint, we have by (19) to maximize at the initial time $t = 0$ the following expression:

$$\begin{aligned} &\max_{\pi, C} E\left[\int_0^T \chi f(C_s, J_s, s) ds + \varepsilon B(V_T^\pi)\right] - \lambda\left(E\left[\int_0^T \chi \hat{C}_s ds + \hat{V}_T^\pi\right] - V_0\right) \\ =\; &\max_{\pi, C} E\left[\chi \int_0^T \left(f(C_s, J_s, s) - \lambda \hat{C}_s\right) ds + \varepsilon B(V_T^\pi) - \lambda\left(\hat{V}_T^\pi - V_0\right)\right]. \end{aligned} \tag{21}$$

Note that we move in (21) the constraint that $\hat{V}^\pi$ has to be fair, under a single, overarching expectation. In some sense, if we are able to maximize the random variable inside the expectation in (21), then we have a candidate for the optimal payout, and thus, a clue for the optimal strategy.

To determine the optimal investment and consumption strategy, one is tempted to fix a time $s \in [0, T]$ and then determine the optimal consumption level at that point in time. With time-additive utility this approach finds the correct characterization of the consumption strategy, see Pennacchi (2008). However, even with time-additive utility this approach ignores that setting consumption at a time t affects the wealth to be invested going forward in time and, therefore, does impact future consumption choices. With Epstein-Zin preferences (recursive utility) these choices are even more intertwined, since future consumption recursively affects current utility levels in addition to consumption levels. To make such a calculation mathematically rigorous requires a suitably defined derivative of the entire consumption path over the time interval $[0,T]$.

In a series of papers Darrel Duffie and coauthors looked into this problem, see Duffie and Epstein (1992a) and Duffie and Skiadas (1994), and used the mathematical concept of Gateaux-derivatives applied to consumption paths to show that these drive the SDF. Section 9.H in Duffie (2001) calculates the Gateaux-derivative explicitly for time-additive utility and shows that the popular characterization of an SDF holds in this case. With stochastic differential utility (Epstein-Zin preferences), Duffie and Skiadas (1994) characterize on pages 125-127 explicitly the SDF, see also Section 9.H together with Appendix F in Duffie (2001). At any point in time their arguments allow us to derive from (21) in our more general setting the following relations for the candidates of the optimal V_T^* and C_s^*:

$$\frac{dB}{dV}(V_T^*) \quad = \quad \frac{\lambda}{\varepsilon V_T^{GP}}, \tag{22}$$

and, as long as $\chi = 1$, for $0 \leq s < T$, and given J_s one gets

$$D_s \frac{\partial f}{\partial C}(C_s^*, J_s, s) \quad = \quad \frac{\lambda}{V_s^{GP}}, \tag{23}$$

where[1]

$$D_s \quad = \quad \exp\left(\int_0^s \frac{\partial f}{\partial J}(C_t^*, J_t, t)dt\right). \tag{24}$$

For simplicity, we write $f'(C,l,s) = \frac{\partial f}{\partial C}(C,l,s)$ and $B'(V) = \frac{dB}{dV}(V)$. Assumption 2 implies that both functions f' and B' are invertible with respect to $C > 0$ and $V > 0$, respectively, and we denote by $f^{\prime,-1}(\cdot,l,s)$ for given (l,s) and by $B^{\prime,-1}(\cdot)$ their respective inverse functions. This allows us to write the candidate for the optimal terminal wealth as

$$V_T^* \quad = \quad B^{\prime,-1}\left(\frac{\lambda}{\varepsilon V_T^{GP}}\right), \tag{25}$$

and the candidate for the optimal consumption at time $s \in [0,T]$ as

$$C_s^* = \chi f^{\prime,-1}\left(\frac{\lambda}{D_s V_s^{GP}}, J_s, s\right). \tag{26}$$

Note that the process D in (24) depends on the utility process J and on the optimal

[1] With recursive preferences, the process D captures the fact that marginal changes in consumption at any time affect (recursively) the entire consumption path and lifetime utility. With time-additive preferences, it can be shown that (23) simplifies to the usual formula, where marginal utility is proportional to an SDF.

consumption process C^*. The key observation is here that the optimal consumption (26) and the terminal wealth characterization in (25), as well as, the process D depend only on the process J and on the GP.

At time $t = 0$, the investor starts with initial wealth $V_0 > 0$. The Lagrange multiplier λ must, therefore, solve equation (20) at time $t = 0$, using the optimal plan (25), (26). The Inada conditions in Assumption 2 let us study the cases $\lambda \to 0$, and $\lambda \to \infty$. This shows us that the right-hand side in (20) runs from ∞ to 0, which ensures the existence of λ.

The well-known martingale technique for time-additive utility, see, e.g. Section 12.4.2 in Pennacchi (2008) or Section 4.4.3 in Cvitanić and Zapatero (2004), as well as its generalization to recursive preferences using the utility gradient technique, see Section 9.H in Duffie (2001), all assume classical no-arbitrage assumptions and end up with the key observation that optimal consumption and terminal wealth can be represented in terms of the classical SDF. Our derivations here generalize this insight in a practically important direction: The SDF is the inverse of the tradeable GP, and our approach has the key advantage that we do not request the restrictive no-arbitrage assumptions of classical finance theory. Furthermore, in our approach the SDF is fully linked to tradeable securities, these are the GP and the baseline security.

As we will show in the next section, these advantages are crucial from a practical point of view, as they have important implications for implementing the optimal strategy and ultimately for employing more realistic long term market models in managing risk than available under classical assumptions.

Let us summarize our findings in (20)-(26) using (19) as follows:

Corollary 6 *Assuming that the optimization problem* (21) *has a unique solution, then the candidates for the benchmarked optimal consumption-savings process* $\hat{V}^*$ *and for the indirect utility process* J *are determined for* $0 \leq t \leq T$ *by the conditional expectations*

$$\hat{V}_t^* = E\left[\chi \int_t^T f^{\prime,-1}\left(\frac{\lambda}{D_s V_s^{GP}}, J_s, s \right) \frac{1}{V_s^{GP}} ds + B^{\prime,-1}\left(\frac{\lambda}{\varepsilon V_T^{GP}} \right) \frac{1}{V_T^{GP}} \middle| \mathcal{A}_t \right], \tag{27}$$

and

$$J_t = E\left[\chi \int_t^T f\left(f^{\prime,-1}\left(\frac{\lambda}{D_s V_s^{GP}}, J_s, s \right), J_s, s \right) ds + \varepsilon B\left(B^{\prime,-1}\left(\frac{\lambda}{\varepsilon V_T^{GP}} \right) \right) \middle| \mathcal{A}_t \right], \tag{28}$$

respectively.

Note that the value on the right hand side of the benchmarked optimal consumption-savings process assumes neither the existence of an equivalent risk-neutral probability measure nor market completeness. The key assumption made is the existence of the GP, which needs trivially to be satisfied to avoid economically meaningful arbitrage.

5 Trading the Optimal Consumption-Savings Process

The previous section found that the optimal consumption decision (at any time) and the terminal wealth depend ultimately *only* on the process characterizing the GP in its given denomination, see equations (25) and (26). Therefore, for further analysis, this section discusses modeling that process.

Under the classical no-arbitrage paradigm, the vector of state variable processes, which models the entire market dynamics, determines, in general, also the optimal solution. Due to the enormous number of state variables that characterize the entire market dynamics, this is not a realistic way of implementing optimal strategies in practice and would leave our previous statements on a purely theoretical level. We have to face, in practice, the impossibility to model and estimate sufficiently accurate the dynamics of all components of the entire global market to implement useful optimal portfolios. This impossibility has been explained, e.g. in DeMiguel et al. (2009) for the closely related task of sample based mean-variance portfolio optimization, and is also argued in Kan et al. (2016), Kan and Zhou (2007), and Okhrin and Schmid (2006).

Instead of aiming for an extremely complex, purely theoretical model for the entire market with unresolvable modeling and parameter estimation challenges, we propose in this paper to exploit the above clarified central role of the GP for the characterization and construction of optimal portfolios. Therefore, it turns out to be extremely useful that proxies of the GP for a given investment universe can be constructed, as demonstrated in Platen and Rendek (2012), and Platen and Rendek (2017). This makes it then practically feasible to approximate well the targeted optimal portfolios and consumption processes.

5.1 Multi-Dimensional Markovian Models

Multi-dimensional, continuous Markovian market dynamics appear to be the class of market models that have been implemented most successfully in the context of utility maximization and derivative pricing. For obtaining tractable optimal strategies we make, therefore, the following assumption:

Assumption 7 *The value* $V_t^{GP} = V^{GP}(t, M_t^1, \ldots, M_t^n)$ *of the discounted GP is a function of a multi-dimensional Markov process* $M = \{M_t = (M_t^1, \ldots, M_t^n)^\top, t \in [0, T]\}$*. The vector* *process M satisfies the SDE*

$$dM_t = \mu_M(t, M_t)dt + \sigma_M(t, M_t)dW_t, \tag{29}$$

where μ_M and σ_M are suitable functions of time t and the Markovian state vector M_t.

As discussed above, our analysis draws our attention to properties of the GP process. It is important to stress that assumption 7 requires the drift vector and volatility matrix to be functions of time and of the vector process M, only. Although state variables may drive the drift and volatility of primary securities, these assumptions mean that no state variable drives the vector process M. In Section 7 we will discuss a model that has this convenient property and matches empirically well the observed GP dynamics.

We assume from now on that we have a tradeable proxy of the GP, which we then identify with the GP for the purposes of this paper. For an example of a construction of a proxy for the GP of the global equity market we refer to Platen and Rendek (2012) and for the GP of developed equity markets to Platen and Rendek (2017). Ultimately, we will show that the optimal consumption-savings process and investment strategy become tradeable in terms of suitable multiple funds and the baseline risk-free security, along the lines of well-established multiple-fund theorems, see e.g. Merton (1971) or Pennacchi (2008). Our analysis in Section 4 postpones the issue of tradeability of optimal solutions but our discussion here now confirms the feasibility of our approach also in this respect.

The characterization of the bequest B and the consumption C in equations (25) and (26) tells us that these are driven by our vector Markov process. Furthermore, due to the characterization in equation (24), the process D can be interpreted as a component of our vector Markov process. Finally, together with (28) and Assumption 7, this implies that the recursive utility J is interpretable as a component of our vector Markov process. Based on equations (23), (27) and (28) we then conclude that the optimal wealth process V^*, the recursive utility process J, the consumption process C^* and the process D are all fully characterized by the current time t together with the current values of the components of the Markov process M.

This allows us to introduce the optimal value (function) $V_t^* = V^*(t, M_t)$ and the life-time utility (function) $J_t = J(t, M_t)$ through equations (27) and (28), as well as the function

$D_t = D(t, M_t)$ through equation (24). This means, for $0 \leq t \leq T$ we have

$$V^*(t,m) = V^{GP}(t,m) E\left[\chi \int_t^T \frac{C^*(s,M_s)}{V^{GP}(s,M_s)} ds + \frac{V^*(T,M_T)}{V^{GP}(T,M_T)} \middle| M_t = m\right], \tag{30}$$

$$J(t,m) = E\left[\chi \int_t^T f\left(C^*(s,M_s), J(s,M_s), s\right) ds + \varepsilon B(V_T^*) \middle| M_t = m\right], \tag{31}$$

where

$$V_T^* = V^*(T, M_T) = B'^{,-1}\left(\frac{\lambda}{\varepsilon V^{GP}(T,M_T)}\right), \tag{32}$$

and, for $0 \leq s \leq T$ we get

$$C^*(s, M_s) = \chi f'^{,-1}\left(\frac{\lambda}{D(s,M_s) V^{GP}(s,M_s)}, J(s,M_s), s\right). \tag{33}$$

Assuming sufficient differentiability of $V^*(\cdot,\cdot)$, an application of Itô's lemma yields the SDE for $V_t^* = V^*(t, M_t) = V^*(t, M_t^1, \ldots, M_t^n)$ in the form

$$dV_t^* = \left(\frac{\partial V^*}{\partial t} + \frac{1}{2}\sum_{i,j=1}^n \sigma_{Mi}\sigma_{Mj}\frac{\partial^2 V^*}{\partial m_i \partial m_j}\right) dt + \sum_{i=1}^n \frac{\partial V^*}{\partial m_i} dM_t^i. \tag{34}$$

The terms in front of dM_t^i in equation (34) yield by standard hedging arguments the following insight:

Corollary 8 (Multiple Fund Separation) *Assume that all components $M^i, i = 1, \ldots, n$ of M are constructed in such a way that they can be traded, giving rise to n risky non-redundant funds plus the baseline security. Then the investor can implement her optimal investment strategy (and fund her consumption-savings profile) by holding at time t*

$$\frac{\partial V^*}{\partial m_i} \tag{35}$$

units of the i-th fund $M_t^i, i = 1, \ldots, n$, and invest the remainder of her wealth in the given baseline security, the locally risk-free security.

Our proof of this result is mathematically similar to those of prior multiple fund separation theorems in the consumption-savings literature, see e.g. Pennacchi (2008). This literature suggests that additional state-variables give rise to additional funds that play a necessary role in optimal investing. Our results, however, clarify this by placing the emphasis fully on the GP denominated in the baseline security and, thus, on the components of M that determine its dynamics. *Only* these components are needed for constructing an optimal investment portfolio and not the many other quantities that characterize the entire financial market. Furthermore, we allow a significantly richer modeling world for capturing more realistically the long-term dynamics of the GP than classical finance theory permits. We request only the existence of the GP and make no longer the additional assumption on the existence of an equivalent risk-neutral probability measure. Since we have discounted our securities by the locally risk-free baseline security, we have to model also its dynamics in our Markovian system, when targeting payoffs that refer to currency units or inflation adjusted payouts, which is a standard task.

It is important to note that in our approach one does not have to care about market incompleteness, i.e. about any additional factors that may be needed to complete the market. All that is needed are the n funds that arise from the n components of M_t. These characterize V_t^* and no other funds are necessary. This reduces the task of portfolio optimization to its core and clarifies the natural inputs that determine the optimal strategy with the GP as central building block.

In addition, our clarification simplifies considerably practical portfolio construction, in particular, when a proxy of the GP is employed. The number of factors needed to model the GP is significantly smaller than the number of state variables characterizing an entire global market model. This becomes most evident, when the uncertainties driving the GP can be captured in a single Brownian motion. In the case of the equity market, the GP is then driven by one source of uncertainty, called the non-diversifiable (systematic) uncertainty. Despite its simplicity, this appears to be a rather realistic case, as forthcoming research will reveal. We will elaborate on a stylized version of this case in Section 7 and then show how this insight simplifies practically relevant applications.

The reader accustomed with optimal investing may be suprised to see that, according to Corollary 8, so-called intertemporal hedge demands do not arise but that only positioning in the terms driving the GP matters. Two important observations must be made: First, we note that it may well be that intertemporal hedge demand show up in other representations of demand that look at positioning in the primary securities instead of positioning in (the driving forces of) the GP. Most important, we note that Assumption 7 does not permit the Markov process M to be driven by state variables that are not spanned by the primary securities. When the vector Markovian process M would be driven by unspanned state variables, on might well obtain some intertemporal hedge demand. While one can easily imagine situations where intertemporal hedge demands apear, the scope of this paper is to focus on practical feasibility and point out practically relevant situations where the analysis can be simplified. As such we are not interested in analyzing situations itself that lead to more complex dynamic asset allocation.

5.2 Scalar Markovian Growth Optimal Portfolio

A particularly important case results when the GP forms a scalar Markov process, that is when we have $n = 1$ and can aggregate its driving uncertainty in a scalar Brownian motion W. Then, we can replace M_t by the GP value V_t^{GP} at time t and write V_t^* as a function of the current GP value, i.e. $V^* = V^*(t, V_t^{GP})$.

Assuming sufficient differentiability of the function V^*, an application of Itô's lemma yields the SDE for $V_t^* = V^*(t, V_t^{GP})$. We then obtain by equation (13) that

$$\begin{aligned} dV_t^* &= \left(\frac{\partial V^*}{\partial t} + \frac{1}{2}\theta_t^2 (V_t^{GP})^2 \frac{\partial^2 V^*}{\partial v^2}\right) dt + \frac{\partial V^*}{\partial v} dV_t^{GP} && (36) \\ &= V_t^*(a_t^V dt + b_t^V dW_t), && (37) \end{aligned}$$

where

$$\begin{aligned} a_t^V V_t^* &= \frac{\partial V^*}{\partial t} + \frac{1}{2}\theta_t^2 \left(V_t^{GP}\right)^2 \frac{\partial^2 V^*}{\partial v^2} + \theta_t^2 V_t^{GP} \frac{\partial V^*}{\partial v}, && (38) \\ \text{and} \quad b_t^V V_t^* &= \theta_t V_t^{GP} \frac{\partial V^*}{\partial v}. && (39) \end{aligned}$$

The term in front of dV_t^{GP} in equation (36) reveals the following result:

Theorem 9 (Two-Fund Separation) *When the discounted GP forms a scalar Markov process, the investor can implement her optimal investment strategy (and fund her optimal consumption-savings profile) by holding at all times*

$$\omega_t = \frac{\partial V^*}{\partial v}(t, V_t^{GP}) \tag{40}$$

units of the (risky) GP, and invest the remainder of her wealth in the given (riskless) baseline security.

It is important to note that in this scalar Markovian case the optimal portfolio can be fully characterized through investment in the GP and the baseline security. This may come as a surprise, as the reader may be accustomed to wealth processes depending on various (untraded or additional) state variables. Here, however, there are no untraded or additional state variables that play any role in the optimal solution, as long as the baseline security is traded, which we assume here. Keeping in mind that our modeling is more general than modeling under classical no-arbitrage assumptions, Theorem 9 generalizes various earlier results in the literature. For example, Pennacchi (2008) reports in his equation (12.70) the wealth weights with unspanned state variables. His equation coincides formally with ours, when all state variables are traded. Yet, our additional observations here are that the scalar Markovianity of the GP simplifies the strategy significantly to an investment into two funds. Note that a similar, slightly more general two-fund separation arises when only one Brownian motion drives a multi-component Markovian SDE that determines the GP as that of one of its components. In this case the equations (36)-(39) become slightly more general when applying the Itô formula to $V^*(t, M_t)$.

6 The Optimal Portfolio Value

The previous section expresses the investment strategy through first-order derivatives of the wealth function, i.e. as a function of the optimal portfolio value. This provides the important insight that it is sufficient to focus primarily on trading the GP and the baseline security. In practical applications this results in a crucial simplification when a proxy of the GP is employed. For potential further theoretical insights but mostly for practical implementations it remains to characterize further the process V^*. To simplify our exposition, we focus now on the case of a scalar Markovian GP, discussed in Subsection 5.2. The handling of more general Markovian multi-component models for characterizing the function $V^*(t, M_t)$ is then straightforward.

The first subsection explains how the general case should be addressed. The following two subsections explain how common preference specifications are covered within our preference framework of Subsection 2.2. In particular, they characterize the investment strategies that result for a scalar Markovian GP dynamics.

6.1 The General Case

The consumption adjusted optimal wealth process $G_t^* = V_t^* + \int_0^t C_s^* ds$ satisfies by equation (37) the SDE $dG_t^* = (V_t^* a_t^V + C_t^*)dt + V_t^* b_t^V dW_t$. This process, when benchmarked, that is $\hat{G}_t = \frac{G_t^*}{V_t^{GP}}$, fulfills the SDE

$$\begin{aligned} d\hat{G}_t &= d\left(G_t^* \frac{1}{V_t^{GP}}\right) = G_t^* d\left(\frac{1}{V_t^{GP}}\right) + \frac{1}{V_t^{GP}} dG_t^* + d < G^*, \frac{1}{V^{GP}} >_t \\ &= \frac{1}{V_t^{GP}} \left(a_t^V V_t^* dt + \chi C_t^* - \theta_t b_t^V V_t^*\right) dt + \left(\frac{V_t^*}{V_t^{GP}} b_t^V - \frac{G_t^*}{V_t^{GP}} \theta_t\right) dW_t. \end{aligned}$$

As mentioned earlier, the process $\hat{G}$ must be a martingale. Consequently, we must have

$$V_t^* \left(a_t^V - \theta_t b_t^V\right) + \chi C_t^* = 0. \tag{41}$$

Together with equations (38) and (39) this allows us to formulate a partial differential equation (PDE) that characterizes $V^*(t,v)$ as follows:

$$\frac{\partial V^*}{\partial t} + \frac{1}{2}\theta_t^2 v^2 \frac{\partial^2 V^*}{\partial v^2} + \chi C^*(t,v) = 0. \tag{42}$$

Furthermore, we have for V_t^* by (13) and (40) the SDE

$$dV_t^* = \left(\theta_t^2 V_t^{GP} \frac{\partial V^*}{\partial v} - \chi C_t^*\right) dt + \theta_t V_t^{GP} \frac{\partial V^*}{\partial v} dW_t = \omega_t dV_t^{GP} - C_t^* dt, \tag{43}$$

see Theorem 9. Since C^* depends on J^*, see equation (33), in general, the PDE (42) needs to be solved jointly with the Hamilton-Jacobi-Bellman (HJB) PDE for J. Equation (3) of Section 9.A in Duffie (2001) reports the HJB equation for a controlled process given in his equation (1). In our setup, equation (42) specifies the controlled process and we find that the HJB equation reads

$$\max_{C^*(t,v)} \left\{ \frac{\partial J^*}{\partial t} + \left(\theta_t^2 \frac{\partial V^*}{\partial v} v - \chi C_t^*\right) \frac{\partial J^*}{\partial v} + \frac{1}{2}\theta_t^2 v^2 \frac{\partial^2 J^*}{\partial v^2} + \chi f\left(C^*, J, t\right) \right\} = 0. \tag{44}$$

When $\chi = 1$ (consumption-savings problem), this means that (formally) the first-order condition yields

$$-\frac{\partial J}{\partial V^*} + f'(C^*, J, t) = 0, \tag{45}$$

where

$$C^*(t,v) = f^{\prime,-1}\left(\frac{\partial J}{\partial V^*}, J, t\right). \tag{46}$$

Using this characterization together with equations (42) and (44) we have to solve the system of PDEs

$$\frac{\partial V^*}{\partial t} + \frac{1}{2}\theta_t^2 v^2 \frac{\partial^2 V^*}{\partial v^2} + \chi f^{\prime,-1}\left(\frac{\partial J^*}{\partial V^*}, J^*, t\right) \quad = \quad 0, \tag{47}$$

$$\begin{aligned} \frac{\partial J^*}{\partial t} + \frac{1}{2}\theta_t^2 v^2 \frac{\partial^2 J^*}{\partial V^{*2}} + \chi f\left(f^{\prime,-1}\left(\frac{\partial J^*}{\partial V^*}, J^*, t\right), J^*, t\right) & \\ + \left(\theta_t^2 \frac{\partial V^*}{\partial v} v - \chi f^{\prime,-1}\left(\frac{\partial J^*}{\partial V^*}, J^*, t\right)\right) \frac{\partial J^*}{\partial V^*} & = 0. \end{aligned} \tag{48}$$

Based on equation (32) and (31) it remains to satisfy also the terminal boundary conditions

$$V^*(T,v) = B'^{,-1}\left(\frac{\lambda}{\varepsilon v}\right) \text{ and } J^*(T,v) = \varepsilon B\left(B'^{,-1}\left(\frac{\lambda}{\varepsilon v}\right)\right), \tag{49}$$

respectively. Finally, to assure the martingale property for $\hat{G}$ we impose the following spatial boundary conditions: For $v \to \infty$ we require $V^*(t,v) \to \infty$ and for $v \to 0$ we require $V^*(t,v) \to 0$.

Note that the solution depends on the Lagrange multiplier λ, i.e. the solution to the PDE (42) is characterized by λ. This parameter should be set to fulfill the initial budget constraint, see our previous discussion in Section 4. Since the characterization of the optimal portfolio value depends strongly on the choice of f, the next subsections discuss particular cases.

6.2 (Time-additive) Preferences over Consumption

In (time-additive) consumption savings problems ($\chi = 1$) with a given utility function u and a rate of time-preference $\delta > 0$ we set[2] $f(c,l,t) = e^{-\delta t}u(c)$. We then obtain $D = 1$ by equation (24) and $D_s \frac{\partial f}{\partial c} = e^{-\delta s}u'(c)$ for the term on the left-hand side in equation (23).

The literature often illustrates time-separable consumption-savings problems with CRRA preferences. Using a (constant) rate of time-preference $\delta > 0$ and a risk-aversion coefficient $0 < \gamma$, one looks typically at $B : x \mapsto e^{-\delta T}x^{1-\gamma}/(1-\gamma)$ and $f : (c,t) \mapsto e^{-\delta t}c^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$. For $\gamma = 1$ the functions are analogous, replacing the power functions by the natural logarithm and leading to $B : x \mapsto e^{-\delta T}\ln(x)$ and $f : (c,t) \mapsto e^{-\delta t}\ln(c)$. Our preference specification covers these cases of CRRA preferences and several more general cases that fulfill Assumption 2.

For further illustration throughout this subsection we discuss exclusively the case of time-additive utility where the investor has CRRA preferences, as described above. We then calculate $f'^{,-1}(c,l,s) = (e^{-\delta s}c)^{-1/\gamma}, B'^{,-1}(x) = (e^{\delta T}x)^{-1/\gamma}$, which gives

$$C_s^* \;=\; (e^{\delta s}\lambda)^{-1/\gamma}(V_s^{GP})^{1/\gamma}, \text{ for } 0 \leq s < T, \text{ and } V_T^* = \varepsilon^{1/\gamma}(e^{\delta T}\lambda)^{-1/\gamma}\left(V_T^{GP}\right)^{1/\gamma}. \tag{50}$$

[2] Alternatively, but more in line with the literature, one may consider the aggregator as $f(c,l,t) = u(c) - \delta l$ to also find that $D_s \frac{\partial f}{\partial c} = e^{-\delta s}u'(c)$. The literature on stochastic differential utility shows that this functional form of the aggregator leads to a J process that is equivalent in terms of preference ranking to the common specification $J_t = \max_{\pi,C} E[\int_t^T \exp(-\delta(T-s))u(C_s)ds|\mathcal{A}_t]$, see Duffie and Epstein (1992a).

This yields

$$\frac{V^*(t,v)}{v} = \frac{1}{\lambda^{1/\gamma}} \int_t^T e^{-\frac{\delta}{\gamma}s} E\left[(V_s^{GP})^{(1/\gamma)-1} \middle| V_t^{GP} = v \right] ds$$
$$+\frac{\varepsilon^{1/\gamma} e^{-\frac{\delta}{\gamma}T}}{\lambda^{1/\gamma}} E\left[(V_T^{GP})^{(1/\gamma)-1} \middle| V_t^{GP} = v \right].$$

We could characterize the consumption-wealth ratio $C_t^*/V^*(t,v)$, but refrain from doing so, since our focus is on investment strategies. The initial budget equation $V_0 = V_0^* = V^*(0, V_0^{GP})$ with $V_0^{GP} = 1$ sets λ, which in turn gives the benchmarked portfolio value as

$$\frac{V^*(t,v)}{v} = V_0 \frac{E\left[\chi \int_t^T e^{-(\delta/\gamma)s} (V_s^{GP})^{1/\gamma-1} ds + \varepsilon e^{-(\delta/\gamma)T} (V_T^{GP})^{1/\gamma-1} \middle| V_t^{GP} = v\right]}{E\left[\chi \int_0^T e^{-(\delta/\gamma)s} (V_s^{GP})^{1/\gamma-1} ds + \varepsilon e^{-(\delta/\gamma)T} (V_T^{GP})^{1/\gamma-1}\right]} \tag{51}$$

$$= V_0 v^{(1/\gamma)-1} \frac{E\left[\chi \int_t^T e^{-(\delta/\gamma)s} \left(\frac{V_s^{GP}}{v}\right)^{1/\gamma-1} ds + \varepsilon e^{-(\delta/\gamma)T} \left(\frac{V_T^{GP}}{v}\right)^{1/\gamma-1} \middle| V_t^{GP} = v\right]}{E\left[\chi \int_0^T e^{-(\delta/\gamma)s} (V_s^{GP})^{1/\gamma-1} ds + \varepsilon e^{-(\delta/\gamma)T} (V_T^{GP})^{1/\gamma-1}\right]}.$$

It is important to note that the fraction on the right-hand side of this representation does not depend on v because of the assumed Markov property for the GP. Taking the derivative shows that

$$\frac{\partial V^*(t,v)}{\partial v} = \frac{1}{\gamma} \frac{V^*(t,v)}{v}.$$

Recall from Theorem 9 that this gives the number of units of the GP to be held.

Specifying the stochastic dynamics of the GP, we could calculate for $t \leq s \leq T$ the expectations $E[(V_s^{GP})^{1/\gamma-1} | V_t^{GP} = v]$ and determine the optimal value function V^*. This would then characterize the optimal investment strategy through Theorem 9. We will discuss this further in the next section, where we consider several dynamics for the GP.

A particularly convenient case is when the optimal value function does not depend on expectations $E[(V_s^{GP})^{1/\gamma-1} | V_t^{GP} = v]$: This corresponds to the case of a log investor ($\gamma = 1$). Then, the above equations simplify to

$$V_t^* = V_0 \frac{\varepsilon\delta + e^{-\delta(t-T)} - 1}{\varepsilon\delta + e^{\delta T} - 1} V_t^{GP}. \tag{52}$$

Additional calculations would recover equations, shown e.g. in Pennacchi (2008).

Equation (52) shows that the optimal value process grows similarly to the GP, where a time-dependent function determines the effect of consumption. This suggests by equation (40) that the investor does not hold all her wealth in the GP. We then calculate $\frac{\partial V^*}{\partial v} = \frac{V_t^*}{v}$

and find based on equations (14) and (39) for a scalar Markovian GP that the investor holds at time $t \in [0, T]$:

$$V_0 \frac{\varepsilon\delta + e^{-\delta(t-T)} - 1}{\varepsilon\delta + e^{\delta T} - 1}$$

units of the GP and the remainder in the locally riskless asset. At the terminal time T the bequest amounts to $V_T^* = V_T^{GP} V_0 \frac{\varepsilon\delta}{\varepsilon\delta + e^{\delta T} - 1}$.

Note that as we increase T, the log-investor holds more and more of her wealth in the GP; taking the limit $T \to \infty$ we find that she holds all her wealth in the GP, which (formally) matches the earlier mentioned insight on the Kelly portfolio.

We emphasize that in the case of a log-investor the optimal strategy is independent of the dynamics of the GP, which is an extremely important observation because the log-utility maximizing portfolio is the one that in the long run generates almost surely the highest value, see Theorem 10.5.1 in Platen and Heath (2010). Thus, an investor who prefers more for less and has an extremely long time horizon is naturally behaving as a log-investor. This type of investor is typically also deeply concerned about the sustainability of our consumption and economic activity, which has become more and more the focus of attention.

6.3 Preferences over Terminal Wealth

In asset allocation problems ($\chi = 0$) it is customary to introduce a rate of time-preference $\delta > 0$, a risk-aversion coefficient $\gamma > 0$, and set $f = 0$, as well as $B : x \mapsto \exp(-\delta T) x^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ and $B : x \mapsto \exp(-\delta T)\ln(x)$ for $\gamma = 1$. Our preference specification covers these and several more general cases of preferences that fulfill Assumption 2.

Preferences over terminal wealth are a special case of our analysis in the previous subsection. For completeness and comparison with the literature, we report the results from equations (50), (51) and (6.2):

$$V_T^* = \varepsilon^{1/\gamma}(e^{\delta T}\lambda)^{-1/\gamma}\left(V_T^{GP}\right)^{1/\gamma} \tag{53}$$

$$\frac{V^*(t,v)}{v} = V_0 v^{(1/\gamma)-1} \frac{E\left[\left(\frac{V_T^{GP}}{v}\right)^{(1/\gamma)-1} \middle| V_t^{GP} = v\right]}{E[(V_T^{GP})^{(1/\gamma)-1}]}, \tag{54}$$

$$\frac{\partial V^*(t,v)}{\partial v} = \frac{1}{\gamma}\frac{V^*(t,v)}{v}. \tag{55}$$

Recall from Theorem 9 that the last term denotes the number of units to be held in the GP. (These results can also be calculated directly, setting $f = 0$ such that $D = 1$ according to equation (24), and calculating $B^{\prime,-1}(x) = (e^{\delta T}x)^{-1/\gamma}$.)

These equations simplify considerably when we assume logarithmic preferences, that is

$\gamma = 1$. In that case we find

$$\frac{V^*(t,v)}{v} = V_0, \tag{56}$$

i.e. the investor holds exclusively the GP, as mentioned already in Subsection 3.3. This is the case where for $T \to \infty$ one obtains almost surely the highest portfolio value. More precisely, on obtains the largest long-term growth rate, that is

$$\lim_{T\to\infty} \ln\left(\frac{V_T^{GP}}{V_0}\right) \geq \lim_{T\to\infty} \ln\left(\frac{V^*(T, V_T^{GP})}{V_0}\right)$$

P-almost surely, by the GP: This key property of the GP makes it so special among all other optimal portfolios and explains intuitively its central role in portfolio optimization.

7 An Empirical Evaluation

The previous Sections introduced our approach based on several assumptions, in particular continuous trading, tradeability of the SDF and the Markov property of its process. This allowed us to come up with convenient consumption-savings and asset allocation strategies that should lead to superior wealth levels. Ultimately, however, it is an empirical question whether our approach does lead to superior wealth levels when implemented in practice. This Section aims to provide support for this.

We study an investor that wants to invest in the US stock market over the long run. The S&P 500 total return index is readily available for the time 1925 to today, even a reconstructed index is provided by R. Shiller for the period 1871 to 1925. Throughout this Section we focus on subperiods of the time from 1925 to 2017 and identify the GP with the S&P 500.

As noted at various stages throughout this paper, preferences with constant relative risk aversion (CRRA) are a common assumptions in financial modeling. Therefore, we also adopt these for empirical evaluation. For simplicity we evaluate terminal wealth, i.e. we set $f = 0, \chi = 0, \varepsilon = 1$ and $B : x \mapsto \exp(-\delta T)x^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$, respectively $B : x \mapsto \exp(-\delta T)\ln(x)$ for $\gamma = 1$. As usual γ refers to the (relative) risk aversion coefficient. We are then in the asset allocation setup of subsection 6.3 and note that the investment strategy is well-specified through the number

$$\frac{\partial V^*(t,v)}{\partial v} = \frac{1}{\gamma}\frac{V^*(t,v)}{v}$$

of units to hold in the S&P 500, see equation (54). For implementations it thus remains to determine the value V^* throughout time and states, which requires us to calculate the (conditional) expectations in equation (54). Throughout we present results only for a relative risk aversion coefficient $\gamma = 3$; this is a popular choice in finance.

The process dynamics of the S&P 500 has been extensively researched and many process specifications in continuous-time have been studied. Throughout this paper we are particularly interested in Markovian processes for the GP with a single Brownian motion. An illustrative and rather realistic case arises when the GP of the stock market is assumed to follow the dynamics of the minimal market model (MMM) of Platen (2001), see also Chapter 13 in Platen and Heath (2010). When using the S&P 500 total return index as a proxy for the GP for the period 1871 to 2017, we obtain $\alpha_0 = 0.1828$ and $\eta = 0.0520$. This allows us to calculate calculate above mentioned expectations $E\left[\left(V_T^{GP}\right)^{(1/\gamma)-1}\middle| V_t^{GP} = v\right]$ for any time t. For completeness, details on the MMM and calculations are provided in Appendix B.2.

We stress that we adopt the MMM process dynamics *only* to allow us the calculation of the (conditional) expectations in equation (54) and the associated portfolio holdings in the S&P 500. Any other Markovian process characterization for the GP would also allow us to proceed analogously. Clearly, the better we capture the stochastic dynamics of the S&P 500, the better our strategy will perform in implementation, however the task of finding the appropriate description is beyond the scope of this paper.

Figure 1 presents the evolution of wealth for an investor starting with initial wealth \$1 from 12/1925 to 1/2015 (violet line) and monthly rebalancing of her portfolio. Her wealth would have grown considerably and increased to approximately \$645, such that we plot wealth on a log scale for easier analysis. To put this into perspective, let us compare this with three alternative strategies. The first is the popular 60/40 strategy for the long run, which invests at all times 60% (40%) of her current wealth in the S&P 500 (the risk-free security). We plot in figure 1 the wealth of our investor following a 60/40 strategy as a red line. Initially, our strategy and the 60/40 strategy follow one another closely, but starting in the 1980‘s our strategy outperfoms the 60/40 strategy. This confirms our strategy which is intended for the long run.

A second comparison is with a monthly rebalancing investment strategy of a investor with mean-variance preferences. It is well-known that her portfolio weight in the risky asset (here the S&P 500) is $\mu_t/(\sigma_t^2 A)$, where μ_t, σ_t denotes the conditional expectation and standard deviation of excess returns over the time period under consideration, as well as A denotes her risk aversion parameter. A straightforward Taylor approximation shows that (in such a first approximation) the CRRA preferences with risk-aversion parameter $\gamma = 3$ the risk-aversion of our mean-variance investor should be set $A = 2\gamma = 6$. We estimate the unconditional

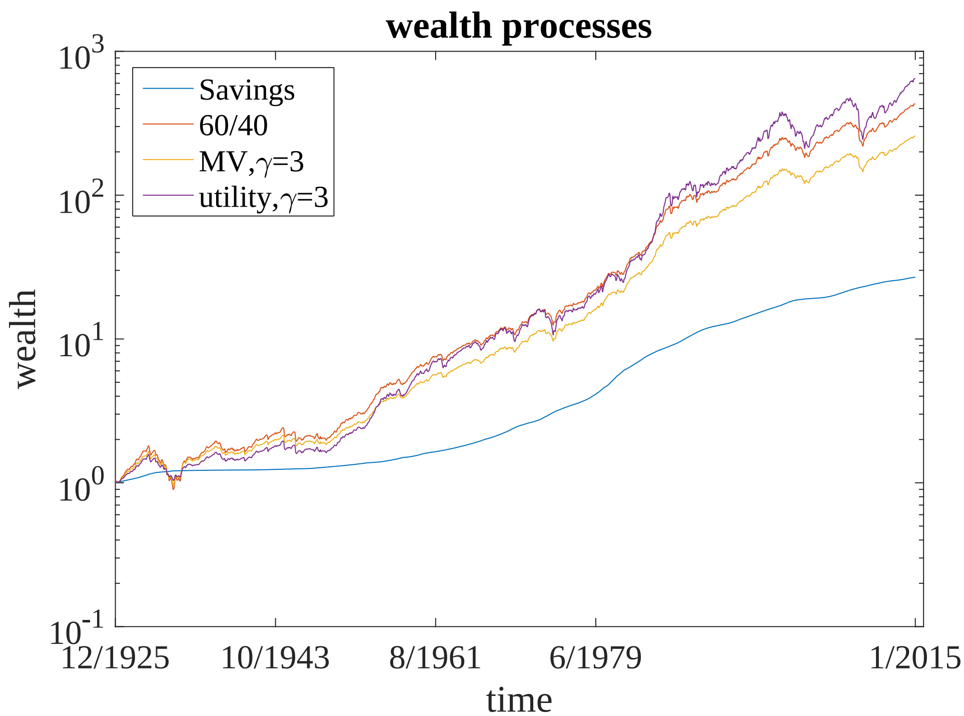

Figure 1: Wealth dynamics for different investment strategies.

expectation and standard deviation of monthly S&P 5000 over the time period and find $\mu = 0.0048, \sigma = 0.0017$. Given the well-known difficulties to implement the mean-variance approach conditionally we adopt these values and implement the mean-variance strategy unconditionally. We plot in figure 1 the wealth of our investor following the associated mean-variance strategy as a yellow line. Wit the exception of the period up to end of the 1940's, the mean-variance strategy outperforms our strategy and underperforms the 60/40 strategy. But our focus is on the long run and here we see that the mean-variance strategy underperforms both our strategy and the 60/40 strategy.

A final last comparison is made with the strategy of investing exclusively in the risk-free security. We plot in figure 1 the wealth of our investor following such strategy as a blue line. As expected, with the exception of the great depression period, this strategy underperfoms all other strategies that we studied above. Clearly, it is not a recommended long term investment strategy.

So far we considered only the situation of an investor that set up her portfolio in 1925. However, it is well known that the performance of investment strategies depends on the starting date and the time horizon of the investor under consideration. Therefore, for completeness we do now investigate how our approach fares against the 60/40 strategy and the mean-variance strategy for different start dates and different time horizons (equivalently

start year	1935	1945	1955	1965	1975	1985	1995	2005	2015
					end year				
Panel (a): Optimal Strategy									
1925	1.4558	1.8159	3.9252	9.4522	14.5106	54.0422	185.4094	372.4709	645.1950
1935		1.1926	2.3215	5.2904	8.2041	29.6918	99.1586	197.5899	340.7004
1945			1.7655	3.7884	5.9432	20.8018	67.1838	132.4785	227.0622
1955				1.8097	2.9427	9.2377	26.2874	49.5946	82.7845
1965					1.6695	4.8120	12.0963	21.6761	34.9976
1975						2.9340	7.5846	13.7611	22.4033
1985							2.3666	4.1214	6.5242
1995								1.6345	2.4674
2005									1.4761
2015									
Panel (b): 60/40 Strategy									
1925	1.6891	2.2470	4.6870	9.7744	15.9909	52.9878	147.5005	267.9764	430.6718
1935		1.3303	2.7748	5.7867	9.4670	31.3701	87.3240	158.6487	254.9684
1945			2.0859	4.3500	7.1166	23.5816	65.6434	119.2599	191.6656
1955				2.0854	3.4117	11.3052	31.4700	57.1742	91.8862
1965					1.6360	5.4211	15.0905	27.4161	44.0611
1975						3.3136	9.2240	16.7581	26.9323
1985							2.7837	5.0573	8.1278
1995								1.8168	2.9198
2005									1.6071
Panel (c): Mean-variance Strategy									
1925	1.4346	1.7769	3.9845	10.0054	12.0157	39.5365	107.2100	175.3230	257.2734
1935		1.2989	2.7891	5.7107	7.3159	27.7153	73.7673	119.3921	192.6267
1945			2.0610	4.1853	5.6807	20.6251	54.4873	88.7298	142.0841
1955				1.9377	3.0152	9.7403	24.5575	39.8849	62.0339
1965					1.7029	5.0673	12.2600	19.9116	30.3380
1975						3.0845	7.7456	12.7960	19.8248
1985							2.4715	4.1097	6.3147
1995								1.6818	2.5571
2005									1.5200

Table 1: Total return varying start and end date.

ending year of the respective portfolio strategy). Table 1 presents the results for the three strategies in three separate Panels. Note that running through the respective diagonals, the reader can easily evaluate results for time horizons from 10 to 90 years in 10 year intervals.

Table 1 reinforces our earlier conclusions based on Figure 1. It shows that our strategy outperforms all other strategies over the long run. In particular, it is better than the other strategies as long as the time horizon is longer than 30-40 years.

8 Conclusion

This paper has studied investing for the long run with a particular focus on practical aspects. This has lead us introduce a stochastic discount factor (SDF) that is more general than the common one. We have stressed the importance of tradeability and identified it as the inverse of the Growth Optimal Portfolio (GP). Using this SDF we have proceeded analogous the martingale technique and characterized the optimal consumption-savings and investment strategy via the SDF. Due to the before-mentioned tradeability this must be tradeable the optimal strategy must be tradeable itself. This has turned our attention to stochastic process properties of the GP and show that fund separation theorems hold as long as the GP is Markovian. Therein, the fund holdings are characterized via partial derivatives of the investor's value function. Finally, we evaluated our strategy empirically by comparing it with the popular 60/40 long term investment strategy and with that of a short-term mean-variance investor. We found that our strategy beats these over the long run.

Appendix: Additional Material

A Recursive Epstein-Zin Preferences

Our setup allows us also to study consumption-savings problems ($\chi = 1$) with preference structures that are more general than time-separable preferences, so-called recursive preferences. The function f is called the (normalized) aggregator of current consumption and continuation utility. A popular form of recursive preferences are the so-called Epstein-Zin preferences, introduced by Epstein and Zin (1989) based on the Kreps-Porteus preference specification. We now discuss these in detail.

A.1 The Aggregator

To describe Epstein-Zin preferences we introduce the, so-called, elasticity of intertemporal substitution parameter $\psi > 0$, the rate of time-preference $\delta > 0$, as well as, a risk-aversion coefficient γ, $0 < \gamma, \gamma \neq 1$. We then define a time-independent function f for strictly positive

$c > 0$ and for strictly positive $l > 0$ (strictly negative $l < 0$) when $\gamma < 1$ (when $\gamma > 1$):

$$f(c,l) = \delta \frac{1-\gamma}{1-\frac{1}{\psi}} l \left(\left(\frac{c}{((1-\gamma)l)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right) \text{ for } \psi \neq 1, \quad \text{(A-1)}$$

$$f(c,l) = \delta(1-\gamma) l \left(\ln(c) - \frac{1}{1-\gamma} \ln\left((1-\gamma)l\right) \right) \text{ for } \psi = 1. \quad \text{(A-2)}$$

Leaving aside the multiplicative term in ε, our optimization problem yields then the Duffie and Epstein (1992a) parametrization of a price taking agent with stochastic differential utility derived from lifetime consumption. This characterizes also a continuous-time version of the Epstein and Zin (1989) preferences that permit separation of risk aversion from the intertemporal rate of substitution. Throughout the current paper, we do allow explicitly for a bequest function $\varepsilon B(V_T^\pi)$. Similarly to Liu (2007), the parameter $\varepsilon > 0$ allows us to adjust the relative importance of bequest and lifetime consumption.

Setting $\psi = 1/\gamma$ in equation (A-1) reduces the above recursive utility consumption-savings problem to a consumption savings problem with time-separable CRRA preferences and (relative) risk aversion coefficient γ. This restriction has been imposed in the literature to compare results with those of the, so-called, Merton consumption-savings problem, see Merton (1971).

To see that the aggregator function f in equations (A-1) and (A-2) fulfills the conditions in Assumption 2 we note first that the function f is twice differentiable on $\mathbb{R}^+$. Taking derivatives based on equation (A-1) ($\psi \neq 1$) and based on equation (A-2) ($\psi = 1$) we obtain:

$$\frac{\partial f}{\partial c} = \delta \left((1-\gamma)l\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} c^{-\frac{1}{\psi}} > 0, \text{ and } \frac{\partial^2 f}{\partial c^2} = -\frac{1}{\psi c} \frac{\partial f}{\partial c} < 0. \quad \text{(A-3)}$$

Based on this representation it is straightforward to check the Inada conditions. We note that Duffie and Lions (1992) and Schroder and Skiadas (1999) provide conditions and proofs for existence and uniqueness of lifetime utility J.

A.2 Optimal Wealth and Consumption-savings Decision

For further illustration throughout this subsection we use the bequest function $B(x) = e^{-\delta T} x^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ and $B(x) = e^{-\delta T} \ln(x)$ for $\gamma = 1$. Equations (A-3) provide the first-order derivatives of the aggregator f that then allows us to derive the inverse w.r.t. consumption. We have $f'^{,-1}(x,l,s) = \delta^\psi \left((1-\gamma)l\right)^{\frac{1-\gamma\psi}{1-\gamma}} x^{-\psi}$ and $B'^{,-1}(x) = e^{-\delta T}/x$. This gives for $0 \leq s < T$ the optimal consumption

$$C_s^* \quad = \quad \delta^\psi \left((1-\gamma)J_s\right)^{\frac{1-\gamma\psi}{1-\gamma}} \lambda^{-\psi}(D_s V_s^{GP})^\psi, \tag{A-4}$$

and terminal value

$$V_T^* = e^{-\delta T}\left(\frac{\varepsilon}{\lambda}V_T^{GP}\right)^{1/\gamma}. \tag{A-5}$$

It allows us to calculate the optimal benchmarked portfolio value function defined in equation (30) as:

$$\begin{aligned} &\frac{V^*(t,v)}{v} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{A-6}) \\ &= E\left[\int_t^T \left((1-\gamma)J_s\right)^{\frac{1-\gamma\psi}{1-\gamma}}\left(\frac{\delta D_s}{\lambda}\right)^\psi (V_s^{GP})^{\psi-1}ds + e^{-\delta T}\left(\frac{\varepsilon}{\lambda}\right)^{\frac{1}{\gamma}}(V_T^{GP})^{\frac{1}{\gamma}-1}\middle| V_t^{GP}=v\right]. \end{aligned}$$

The Lagrange multiplier λ follows from solving the initial budget equation $V_0 = V_0^* = V^*(0, F_0)$. This provides a full characterization of the optimal value process.

If we assume the intertemporal elasticity of substitution as $\psi = 1/\gamma$, then we have $1-\gamma\psi = 0$ such that $((1-\gamma)J_s)^{\frac{1-\gamma\psi}{1-\gamma}} = 1$ and the process J does no longer play a role in equation (A-6). We are then back in the representation presented in the previous subsection with the value function (51). This is as expected, since it is well-known that the case $\psi = 1/\gamma$ corresponds to time-additive CRRA preferences.

The case without bequest ($\varepsilon = 0$) allows us to further simplify this result. In that case we can factor out $\delta^\psi\lambda^{-\psi}(1-\gamma)^{(1-\gamma\psi)/(1-\gamma)}$ and using $V_0 = V^*(0, V_0^{GP})$ we find based on (A-6) that at all times $0 \leq t \leq T$ one has

$$V^*(t,v) = V_0 v \frac{E\left[\int_t^T J_s^{\frac{1-\gamma\psi}{1-\gamma}} D_s^\psi (V_s^{GP})^{\psi-1}ds \middle| V_t^{GP}=v\right]}{E\left[\int_0^T J_s^{\frac{1-\gamma\psi}{1-\gamma}} D_s^\psi (V_s^{GP})^{\psi-1}\right]}. \tag{A-7}$$

If we further assume the intertemporal elasticity of substitution as $\psi = 1$, then the function V^* simplifies to

$$V^*(t,v) \quad = \quad V_0 v \frac{E\left[\int_t^T J_s D_s ds \middle| V_t^{GP}=v\right]}{E\left[\int_0^T J_s D_s\right]}.$$

Further analysis of this equation and of (A-7) requires among others studying the distributional properties of the GP.

B Optimal Portfolio Value for Particular GP Processes

Section 3 identifies properties of the GP that would allow us to find the minimal possible price for a targeted payoff, while subsequent sections show how to hedge this payoff conveniently by only investing in a proxy of the GP and the baseline security. This lead us in Section 5 to study tradeable proxies of the GP.

In addition, from a modeling perspective, this draws our attention to properties of the GP and modeling its dynamics adequately. Constructing a proxy of the GP and interpreting it as GP would then avoid the practically challenging and almost impossible task of modeling and estimating all those factors and parameters that determine the entire market dynamics and that, in principle, one needs to have access to in order to identify accurately the theoretically precise GP. Therefore, we will assume in the following that we have constructed a tradeable proxy of the GP and, therefore focus on exploring properties of the GP that are sufficient for hedging in consumption-savings investments.

It is important to recall that different to the well-known martingale technique, see Pennacchi (2008), and Cvitanić and Zapatero (2004), we do not assume the existence of an equivalent risk neutral probability measure. This allows us to use a much richer modeling world when characterizing the dynamics of the market. The first subsection below assumes a classical market model with a constant investment opportunity, whereas the second subsection considers a market model that is more realistic and does not fit any longer into the world of classical market models.

B.1 Constant Investment Opportunity Set

A most convenient case has been widely studied in the literature, where one assumes a Black-Scholes dynamics with constant volatility for the GP. When we assume a constant market price of risk $\theta_t = \theta > 0$, we obtain for any $0 \leq t \leq s \leq T$ for the discounted GP the expression

$$\begin{aligned} V_s^{GP} &= V_t^{GP} \exp\left\{ \frac{\theta^2}{2}(s-t) + \theta(W_s - W_t) \right\} \\ \text{and so } E\left[\left(V_s^{GP}\right)^{(1/\gamma)-1} \middle| V_t^{GP} \right] &= \left(V_t^{GP}\right)^{(1/\gamma)-1} \exp\left\{ \theta^2 \frac{1-\gamma}{2\gamma^2}(s-t) \right\}. \end{aligned}$$

This allows us to express the value function V^* in terms of the GP as in Subsection 5.2. We then derive in the case of preferences over terminal wealth based on (54) the benchmarked value function

$$\hat{V}_t^* = \frac{V^*(t, V_t^{GP})}{V_t^{GP}} = V_0 \exp\left(\theta^2 \frac{\gamma - 1}{2\gamma^2} t\right) \left(V_t^{GP}\right)^{(1/\gamma)-1} . \tag{B-8}$$

Multiplying this expectation through with $V_t^{GP} = v$, yields $V^*(t, V_t^{GP}$. By taking the first order derivative of $V^*(t, V_t^{GP})$ w.r.t. v, we note that the right hand side in this equation is proportional to $V^*(t, V_t^{GP}$, that is

$$\frac{\partial V^*(t, V_t^{GP})}{\partial V_t^{GP}} = \frac{1}{\gamma} \frac{V^*(t, V_t^{GP})}{v} = \frac{1}{\gamma} \hat{V}_t^*.$$

According to Theorem 9, therefore, $V^*(t, V_t^{GP})$ also denotes γ times the number of units of the GP the investor holds.

Next, we look at the case of time-additive CRRA preferences, where we write

$$\rho = \theta^2 \frac{1 - \gamma}{2\gamma^2} - \delta/\gamma.$$

We derive that

$$\begin{aligned}
& E\left[\int_t^T e^{-(\delta/\gamma)s} (V_s^{GP})^{1/\gamma - 1} ds + \varepsilon e^{-(\delta/\gamma)T} (V_T^{GP})^{1/\gamma - 1} \middle| V_t^{GP} = v\right] \\
= \; & \left(V_t^{GP}\right)^{1/\gamma - 1} e^{-(\delta/\gamma)t} \int_t^T \exp\left\{\rho(s - t)\right\} ds + \varepsilon \left(V_t^{GP}\right)^{1/\gamma - 1} e^{-(\delta/\gamma)t} \exp\left\{\rho(T - t)\right\} \\
= \; & \left(V_t^{GP}\right)^{1/\gamma - 1} e^{-(\delta/\gamma)t} \frac{1}{\rho} \left(\exp\left\{\rho(T - t)\right\} - 1\right) + \varepsilon \left(V_t^{GP}\right)^{1/\gamma - 1} e^{-(\delta/\gamma)t} \exp\left\{\rho(T - t)\right\}
\end{aligned}$$

In the case of time-additive CRRA preferences we then find based on (51) that

$$\frac{V^*(t, v)}{v} = V_0 v^{1/\gamma - 1} e^{-(\delta/\gamma)t} \cdot \frac{\frac{1}{\rho}\left(\exp\left\{\rho(T - t)\right\} - 1\right) + \varepsilon \exp\left\{\rho(T - t)\right\}}{\frac{1}{\rho}\left(\exp\left\{\rho T\right\} - 1\right) + \varepsilon \exp\left\{\rho T\right\}} .$$

B.2 Minimal Market Model

In the Minimal Market Model (MMM) of Platen (2001), see also Chapter 13 in Platen and Heath (2010), the (discounted) GP V_t^{GP} can be expressed as the product

$$V_t^{GP} = Y_t \alpha_t \tag{B-9}$$

of a square root process $(Y_s)_{0 \le s \le T}$ of dimension four with an exponential function of time $\alpha_t = \alpha_0 \exp(\eta t), \eta > 0, \alpha_0 > 0$, where

$$dY_t = (1 - Y_t)\eta dt + \sqrt{\eta Y_t} d\tilde{W}_t \tag{B-10}$$

for $0 \leq t < \infty, Y_0 = \frac{1}{\alpha_0}$, with $\tilde{W}$ denoting a Brownian motion. The volatility of V_t^{GP} equals then that of Y_t, which is

$$\tilde{b}_t^F = \sqrt{\frac{\kappa}{Y_t}} = \sqrt{\frac{\kappa\alpha_t}{V_t^{GP}}}. \tag{B-11}$$

The only two parameters needed are $\alpha_0 > 0$ and η. Both can be fitted by noting that the quadratic variation of $\sqrt{V_t^{GP}}$ equals

$$\varphi(t) =< \sqrt{V_t^{GP}} >_t= \frac{\alpha}{4}\left(e^{\eta t} - 1\right), \tag{B-12}$$

such that

$$\eta = \frac{1}{t}\ln\left(\frac{< \sqrt{V_t^{GP}} >_t}{\alpha_0} + 1\right), \tag{B-13}$$

which allows one to estimate η and also α_0 for sufficiently long time periods.

As shown in Platen and Heath (2010), the MMM does not fit under the classical no-arbitrage paradigm because the inverse of the discounted GP is not a true martingale and only a strict local martingale. Therefore, the density of the putative risk-neutral measure $\frac{V_0^{GP}}{V_t^{GP}}$ is only a strict local martingale, and an equivalent risk-neutral probability measure does not exist. From equation (8.7.14) in Platen and Heath (2010) it follows for $\gamma \in (1, \infty]$ and $0 \leq t \leq s < \infty$ that

$$\begin{aligned} & E[(V_s^{GP})^{(1/\gamma)-1}|V_t^{GP}] \\ = \; & (2(\varphi(s) - \varphi(t))^{(1/\gamma)-1} \exp\left\{-\frac{V_t^{GP}}{2(\varphi(s) - \varphi(t))}\right\} \sum_{k=0}^{\infty} \left(\frac{V_t^{GP}}{2(\varphi(s) - \varphi(t))}\right)^k \frac{\Gamma(\frac{1}{\gamma} + 1 + k)}{k!\;\Gamma(k+2)}, \end{aligned} \tag{B-14}$$

where

$$\varphi(t) \;\; = \;\; \frac{\alpha_0}{4\eta}\left(e^{\eta t} - 1\right), \tag{B-15}$$

and Γ denotes the Gamma function. For instance, in the special case $\gamma = 1/2$ we have

$$E[V_s^{GP}|V_t^{GP}] = 4(\varphi(s) - \varphi(t)) + V_t^{GP}. \tag{B-16}$$

In the case $\gamma = 1$ we have log-utility and obtain

$$E[(V_s^{GP})^0|V_t^{GP}] = 1$$

which yields an important simplification of any calculation.

References

Bai, J., Ng, S., 2002. Determining the number of factors in approximate factor models. Econometrica 70 (1), 191–221.

Baldeaux, J., Grasselli, M., Platen, E., 2015. Pricing currency derivatives under the benchmark approach. Journal of Banking and Finance 53, 34–48.

Campbell, J. Y., Viceira, L., 1999. Consumption and portfolio decisions when expected returns are time-varying. Quarterly Journal of Economics 114, 433–495.

Campbell, J. Y., Viceira, L., 2002. Strategic Asset Allocation: Portfolio Choice for Long-Term Investors. Clarendon Lectures in Economics. Oxford University Press.

Chacko, G., Viceira, L. M., 2005. Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. Review of Financial Studies 18 (4), 1369–1402.

Christensen, M. M., 2012. On the history of the growth optimal portfolio. In: Gyorfi, L., Ottucsak, G., Walk, H. (Eds.), Machine Learning for Financial Engineering. Vol. 8 of Advances in Computer Science and Engineering. World Scientific Publishing Corporation, Ch. 1, pp. 1–79.

Cochrane, J. H., 2001. Asset Pricing. Princeton University Press.

Cox, J. C., Huang, C. F., 1989. Optimum consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory 49, 33–83.

Cvitanić, J., Zapatero, F., 2004. Introduction to the economics and mathematics of financial markets. MIT Press.

Davis, M. H., Lleo, S., 2015. Risk-Sensitive Investment Management. World Scientific Publishing Corporation.

Delbaen, F., Schachermayer, W., 2006. The Mathematics of Arbitrage. Springer Finance.

DeMiguel, V., Garlappi, L., Uppal, R., 2009. Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? Review of Financial Studies 22 (5), 1915–1953.

Du, K., Platen, E., 2016. Benchmarked risk minimization. Mathematical Finance 26 (3), 617–637.

Duffie, D., 2001. Dynamic Asset Pricing Theory, 3rd Edition. Princeton University Press.

Duffie, D., Epstein, L. G., 1992a. Asset pricing with stochastic differential utility. Review of Financial Studies 5 (2), 411–436.

Duffie, D., Epstein, L. G., 1992b. Stochastic differential utility. Econometrica 60 (2), 353–394.

Duffie, D., Lions, P., 1992. Pde solutions of stochastic differential utility. Journal of Mathematical Economics 21, 577–606.

Duffie, D., Skiadas, C., 1994. Continuous-time security pricing – a utility gradient approach. Journal of Mathematical Economics 23, 107–131.

Epstein, L., Zin, S., 1989. Substitution, risk aversion and the temporal behaviour of consumption and asset returns: A theoretical framework. Econometrica 57, 937–969.

Fernholz, D., Karatzas, I., 2010. Probabilistic aspects of arbitrage. In: Chiarella, C., Novikov, A. (Eds.), Contemporary Quantitative Finance. Springer Verlag, pp. 1–17.

Harrison, J. M., Kreps, D. M., 1979. Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20, 381–408.

Kan, R., Wang, X., Zhou, G., 2016. On the value of portfolio optimization in the presence of estimation risk: the case with and without risk-free asset. Working Paper, University of Toronto.

Kan, R., Zhou, G., 2007. Optimal portfolio choice with parameter uncertainty. Journal of Financial and Quantitative Analysis 42 (3), 621–656.

Karatzas, I., Kardaras, C., 2007. The numéraire portfolio in semimartingale financial models. Finance and Stochastics 11, 447–493.

Kelly, J., 1956. A new interpretation of information rate. Bell Systems Technical Journal 35 (4), 917–926.

Kraft, H., Seifried, F. T., Steffensen, M., 2013. Consumption-portfolio optimization with recursive utility in incomplete markets. Finance and Stochastics 17, 161–196.

Liu, J., 2007. Portfolio choice in stochastic environments. Review of Financial Studies 20 (1), 1–38.

Loewenstein, M., Willard, G., 2000. Local martingales, arbitrage, and viability: Free snacks and cheap thrills. Econometric Theory 16 (1), 135–161.

Long, J., 1990. The numéraire portfolio in semimartingale financial models. Journal of Financial Economics 26, 29–69.

Ludvigson, S., Ng, S., 2007. The empirical riskreturn relation: A factor analysis approach. Journal of Financial Economics 83 (1), 171–222.

MacLean, L., Thorp, E., Ziemba, W., 2010. The Kelly Capital Growth Investment Criterion: Theory and Practice. World Scientific Publishing Corporation.

Merton, R. C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 371–413.

Okhrin, Y., Schmid, W., 2006. Distributional properties of portfolio weights. Journal of Econometrics 134 (1), 235–256.

Pennacchi, G., 2008. Theory of Asset Pricing. Pearson Addison Wesley.

Platen, E., 2001. A minimal financial market model. Trends in Mathematics. Birkhauser, pp. 293–301.

Platen, E., 2002. Arbitrage in continuous complete markets. Advances in Applied Probability 34 (3), 540–558.

Platen, E., 2006. A benchmark approach to finance. Mathematical Finance 16 (1), 131–151.

Platen, E., Heath, D., 2010. A Benchmark Approach to Quantitative Finance. Springer Verlag.

Platen, E., Rendek, R., 2012. Approximating the numeraire portfolio by naive diversification. Journal of Asset Management 13 (1), 34–50.

Platen, E., Rendek, R., 2017. Market efficiency and the growth optimal portfolio for equities. Working Paper, University of Technology Sydney.

Schroder, M., Skiadas, C., 1999. Optimal consumption and portfolio selection with stochastic differential utility. Journal of Economic Theory 89, 68–126.