# Stochastic Volatility and the (Equilibrium) Discount Function*

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**Abstract**

This paper studies the stochastic discount factor as a function of aggregate endowment, the so-called discount function. Our analysis starts with economic fundamentals (endowment, firm book value) that are driven by a bivariate geometric Brownian motion. We derive the properties of the discount function, which lead to stochastic volatility of a stochastic firm price process. We then introduce investors and characterize the functional structure of the representative agent that supports equilibrium prices. Here marginal utility must be that of the equilibrium stochastic discount function and we conclude that volatility is stochastic in our setup if, and only if, investors are heterogeneous in their risk preferences. This provides a risk-sharing explanation for the stochastic nature of volatility of asset prices. Our results tie together different strands of the discrete- and continuous-time option pricing literature, and of the literature on stochastic discount factors (representative agents).

**Keywords**

stochastic volatility, state-dependent volatility, heterogeneity

**JEL Classification**

D51, D53, G13

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# 1 Introduction

Empirical studies document extensively that the volatility of asset returns[1] evolves randomly over time; the theoretical literature attributes this to a variety of explanations (stochastic nature of dividends, of the information process, heterogeneous beliefs). A foundation of financial economics is the stochastic discount factor (a.k.a. pricing kernel); one may expect stochastic volatility of prices to reflect itself in the stochastic discount factor but this has not been studied, so far. Our goal is to study its properties and how these are derived from economic fundamentals.

The paper studies a claim on terminal firm value in a continuous-time version of the discrete-time economy of Rubinstein (1976). The underlying price processes of aggregate endowment and firm value follow a two-dimensional (bivariate) geometric Brownian, i.e. volatility of the underlying economic variables is *constant*. The aggregate endowment drives the state-price process (pricing kernel) and enters into the price process of the firm claim.

Our analysis proceeds in two steps. In the first step we study structural properties of the state-price process and relate stochastic volatility to a power function of the aggregate endowment and equivalently to a CRRA type preference structure of the so-called (price supporting) representative agent. In the second step we study in detail aggregation and relate that CRRA property to homogeneity/heterogeneity in risk-preferences of two agents with CRRA preferences. This allows us to conclude here that heterogeneity in risk preferences leads to stochastic volatility.

Our paper contributes in many ways to the literature. Our *first contribution* consists in tieing together insights from the single-period pricing literature with those from the continuous-time pricing literature[2]. First of all, starting with Heston (1993), the continuous-time option pricing literature showed that stochastic volatility of the underlying stock price process leads to the so-called implied volatility "smile" (a stylized fact of options prices).

[1]Volatility describes the standard deviation of stochastic returns for financial assets. Stochastic volatility means that volatility is driven by a process in addition to the process that drives stock returns; it is different from time-dependent and state-dependent volatility.

[2]We focus here on options on firm value. Analogously, we could study options on terminal endowment; for simplicity we do not work out the details throughout this paper. From an econometrician's viewpoint, our analysis links to latent variables in state-price processes, see, e.g. Chabi-Yo et al. (2007).

Second, Merton (1973) and Rubinstein (1976) proved in single-period models that the Black-Scholes formula holds, when CRRA agents are homogeneous in their risk-preferences; Benninga and Mayshar (2000) showed the converse, i.e. heterogeneity in CRRA leads to the "smile" pattern.

Our *second contribution* consists in clarifying the structure of the representative agent. It is well known, see Rubinstein (1974) and Constantinides (1982), that equilibrium prices can be represented through a representative agent, i.e. a representative agent can be defined that leads to equilibrium prices. These types of analysis ignore that the price-setting representative agent should not just support the prices over a given fixed time-period, but that it should support the entire price dynamics; put differently, this ignores that the functional form of the representative agent may depend on the time-horizon under consideration. When volatility is stochastic, our analysis implies that heterogeneous CRRA agents *cannot* aggregate to a representative agent with CRRA preferences. Our paper thereby highlights the importance of being careful with assumptions about the utility functions for representative agents and to consider temporal consistency.

Finally, our *third and most important contribution* consists in broadening our understanding of the stochastic nature of volatility. The literature attributes stochastic volatility to the stochastic nature of dividends in representative agent economies, see, e.g. Cochrane et al. (2008), Bhamra and Uppal (2011) and references therein, to the stochastic nature of the information process (Admati (1985), Brock and Hommes (1997)), and to the impact of heterogeneous beliefs, see, e.g., Detemple and Murthy (1994), Zapatero (1998) and Li (2007). Our paper, provides an alternative explanation related to heterogeneity: here, we attribute stochastic volatility to the aggregation of heterogeneous risk-preferences.

In the next section we introduce the process structure of the underlying economic variables, the firm claim and the so-called representative agent. The third section links the functional form of the discount function to stochastic volatility of the firm price processes. In the fourth section we study equilibrium in an exchange economy where agents have CRRA preferences and link stochastic volatility to the utility of the representative agent and ultimately to heterogeneity in agent's risk preferences. The paper concludes with section 5. All proofs are postponed to the appendix.

## 2 Setup

This section introduces the main objects of study: the pricing system, the discount function, and the firm claims. Throughout this paper we assume there is a single perishable consumption good and all units are expressed in terms of that good. Time 0 is today; at time $T > 0$ all economic activity ceases.

### 2.1 State-price Processes and Functions

There is a long tradition to study price processes in terms of aggregate consumption; typically, a representative agent with an exogenously specified utility function is invoked and, in addition, the aggregate consumption stream is exogenously specified. Within such analysis there is no distinction between aggregate endowment and aggregate consumption. At this stage, however, we do not want to depend on a specification of agents' preferences or consumption.

We only specify exogenously the so-called aggregated endowment process $Y$, observable at all times: we assume $Y_0 = 1$ and that it evolves over time $0 \leq t \leq T$ according to the stochastic differential equation

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y; \tag{1}$$

here, $\mu_Y$ and $\sigma_Y > 0$ are constants that are known to agents and $(W_t^Y)_t$ denotes a standard Wiener process on a suitable probability space $(\Omega, \mathcal{F}, P)$.

This process is also known as geometric Brownian motion with constant volatility $\sigma_Y$. Therefore, we believe this is a good starting point to study the impact of heterogeneity on price processes.

It is well known, see e.g. Duffie (2001) that an arbitrage-free linear pricing system can be supported by a *state-price process* $(M_t)_{0\leq\tau\leq T}$, i.e. the time $t$ price of a claim to any Markovian payout process $(\Pi_\tau)_{t\leq\tau\leq T}$ of the consumption good is

$$\frac{1}{M_t} E\left[\int_t^T \Pi_\tau M_\tau d\tau \middle| \Pi_t, Y_t\right]. \tag{2}$$

We assume that the market is (dynamically) complete in claims on the aggregate consumption stream. This is warranted, e.g., when a riskfree security and a claim on the endowment can be traded. Then, the state-price process is unique.

Throughout this paper we study the following class of state-price processes that allows us to separate the impact of time and aggregate endowment:

**Definition 1 (Time-separable State-price Process)** [3]

*A* state-price function (SPF, a.k.a. discount function) *is a strictly positive function* $m$ *on the positive real line with the property that* $P$*-a.s.* $M_t = \gamma(t) \cdot m(Y_t)$ *for all* $0 \leq t \leq T$ *for a suitable, strictly positive function* $\gamma$ *on the time interval* $[0, T]$, *and subject to "appropriate" technical conditions.*

## 2.2 The Firm Price

Throughout, we study a claim on an asset that is (partially) correlated with the aggregate endowment. We refer to the underlying asset as the firm book value and denote it by $F$. The current book value $F$ is observable at all times and evolves over time $0 \leq t \leq T$ according to the stochastic differential equation (a.k.a. geometric Brownian motion)

$$dF_t = \mu_F F_t dt + \sigma_F F_t dW_t^F, \tag{3}$$

where $(W_t^F)_t$ is standard Wiener processes on the probability space $(\Omega, \mathcal{F}, P)$, $F_0 = 1$, and $\mu_F$ as well as $\sigma_F > 0$ are constants. The instantaneous correlation between the Wiener processes $W^F$ and $W^Y$ is constant and we denote it by $-1 \leq \kappa \leq 1$. Similar to the aggregate endowment process $Y$ we model the firm book value $F$ through geometric Brownian motion and exclude stochastic volatility in fundamentals.

For simplicity, we focus on a claim that pays only at time $T$ the book value of the firm; nothing[4] is paid before time $T$. We refer to the *shadow price* $S_t$ of this claim on firm book value as the *firm price* at time $t$; this defines a stochastic process $S = (S_t)_{0 \leq t \leq T}$. The firm price is a function of current time $t$, book value $F_t$ and endowment $Y_t$, i.e. $S_t = S(t, F_t, Y_t)$.

[3]The "appropriate" technical conditions ensure sufficient differentiability and that expectation/differentiation can be exchanged. We refrain from working this out.

[4]A claim that pays a stream of the firm book value at *all* times could be valued using the state-price process. Studying such a claim would be more sensible but would also considerably complicate our analysis. To illustrate our qualitative insights we focus on a claim that pays only at a single date.

The pricing property of the state-price process, equation (2), implies that

$$S(t,f,y) = f\frac{\gamma(T)}{\gamma(t)}\frac{E\left[m(yZ_{tT}^{Y})Z_{tT}^{F}\right]}{m(y)}, \tag{4}$$

where

$$\ln Z_{tT}^{Y} = \left(\mu_Y - \frac{\sigma_Y^2}{2}\right)(T-t) + \sigma_Y\sqrt{T-t}U^Y,\ \ln Z_{tT}^{F} = \left(\mu_F - \frac{\sigma_F^2}{2}\right)(T-t) + \sigma_F\sqrt{T-t}U^F,$$

and $U^Y, U^F$ are standard normal random variables that have correlation $\kappa$ with each other. Note that $Y_T \overset{d}{=} Y_t Z_{tT}^Y$ and $F_T \overset{d}{=} F_t Z_{tT}^F$, where $\overset{d}{=}$ denotes equality in distribution.

For future reference throughout the remainder of this paper we denote by $\epsilon_S$ the elasticity of the firm price w.r.t. the current aggregate endowment, i.e.

$$\epsilon_S(t,f,y) = \frac{\partial S}{\partial y}\frac{y}{S_F}. \tag{5}$$

Equation (4) shows that $\frac{\partial S}{\partial f}$ is linear in $f$; therefore, $\epsilon_S$ does not depend on current book value $f$ and for simplicity of exposition we drop the dependence of $\epsilon_S$ on firm (book) value $f$ throughout this paper. In addition, we define a function $V_S$ by setting

$$V_S(t,y) = \sigma_F^2(1-\kappa^2) + \left(\sigma_Y\epsilon_S(t,y) + \kappa\sigma_F\right)^2. \tag{6}$$

# 3 The Volatility Process

This section derives the volatility price process of the firm claim and studies its properties. We are particularly interested in the properties of the volatility process and the functional form of the SDF, equivalently of the structure of the relative risk aversion function $\bar{\rho}$ of the representative agent. (The next section will study equilibrium and characterize $\bar{\rho}$ based on fundamentals of an exchange economy.)

## 3.1 Characterization

Itô's lemma shows that the firm price dynamics is

$$dS_t \;=\; \mu_S(t,F_t,Y_t)S_{Ft}dt + \frac{\partial S}{\partial f}\sigma_F F_t dW_t^F + \frac{\partial S}{\partial y}\sigma_Y Y_t dW_t^Y, \tag{7}$$

for a suitable drift function $\mu_S$. Rewriting the description in equation (7), the Appendix shows:

**Theorem 2** *There is a standard Wiener process $\tilde{W}^F$ and a function $\mu_F$ such that the dynamics of the firm price process is $dS_t = \mu_S(t, F_t, Y_t) S_t dt + \sqrt{V_S(t, Y_t)} S_t d\tilde{W}^F_t$;*

Because $\tilde{W}^F$ is a standard Wiener processes, the term $V_S(t, Y_t)$ captures what is commonly referred as volatility[5]; note that it is a (stochastic) process driven by the aggregate endowment. There are two contributions to stock price volatility $V_S$: the volatility of book value enters through a constant level $\sigma_F$, because the firm price $S$ depends linearly on firm book value, see equation (4). The second contribution to volatility is through the aggregate endowment via $(\sigma_Y \epsilon_S + \kappa \sigma_F)^2$; here, the current endowment $y = Y_t$ enters (indirectly) through the elasticity $\epsilon_S$; therefore, this will be our main object of study.

Itô's lemma allows us to characterize the dynamics of volatility $V_{St} = V_S(t, Y_t)$ as a stochastic process:

$$dV_{St} = \left( \frac{\partial V_S}{\partial y} + \frac{1}{2} \frac{\partial^2 V_S}{\partial y^2} \right) dt + \sigma_Y \left( \frac{\partial V_S}{\partial y} Y_t \right) dW^Y_t. \tag{8}$$

Our focus is on the second term, in particular on the function $y \mapsto \frac{\partial V_S}{\partial y} y$. The literature distinguishes three forms of non-constant volatility: (1) time-dependent volatility where volatility is *only* a function of time; (2) state-dependent volatility, where the stochastic movement of volatility over time that is driven by the underlying asset; (3) stochastic volatility, where the stochastic movement of volatility over time that is *not* driven by the underlying asset, i.e. it must be driven by a second stochastic process. The time-dependence of $\epsilon_S$ may lead to time-varying volatility, but firm price volatility is varying stochastically[6] only if $\epsilon_S$ is *not* constant in $y$. To define *stochastic volatility* we focus on the dependence of $\epsilon_S$ on the aggregated endowment. This leads us naturally to the following definition of stochastic volatility:

**Definition 3** [7] *We say that volatility of the firm price process is* stochastic, *if there is a time point $0 \leq t < T$ and an open interval such that the function $\epsilon_S(t, \cdot)$ is not constant on that interval. Otherwise, we say that firm volatility is* not stochastic.

[5]Formally, $V_S$ is variance, but it is a common convention to refer to it as volatility.

[6]The term $\epsilon_S$ does not depend on firm book value and so the firm price cannot exhibit state-dependent volatility.

[7]The definition is a dichotomy; note that for volatility that is not stochastic, the elasticity $\epsilon_S$ and thus the volatility $V_S$ can only depend on time $t$.

## 3.2 The Discount Function and the Representative Agent's Risk-aversion

The state-price function $m$ and thus the pricing system can be supported by a representative agent with a suitable utility function $u_r$, i.e. there exists a function $u_r$ with the property that $m = \lambda u_r'$ for a suitable constant $\lambda$. We denote by $\bar{\rho}$ the relative risk aversion function of the representative agent.

In financial economics, it is common to postulate that the representative agent has preferences that exhibit constant relative risk aversion, i.e. the relative risk aversion function $\bar{\rho}$ is constant; this means that the representative agent's utility function $u_r$ is a power function on the entire positive real line, i.e. there exist suitable[8] constants $\alpha, \beta, \lambda$ such that the state-price function $m$ and the function $u_r$ are for all $y > 0$

$$m(y) = \alpha y^{\beta}, \text{ or equivalently } u_r(y) = \frac{\alpha}{(\beta+1)\lambda} y^{\beta+1} + const.$$

We calculate using well-known formulas for (bivariate) lognormal random variables:

$$\begin{aligned} S(t,f,y) &= f\frac{\gamma(T)}{\gamma(t)} E\left[Z_{tT}^{F}\left(Z_{tT}^{Y}\right)^{\beta}\right] \\ &= f\frac{\gamma(T)}{\gamma(t)} \exp\left(\left(\mu_F + \beta\mu_Y + \frac{\sigma_Y^2}{2}\beta(\beta+1) + 2\beta\kappa\sigma_F\sigma_Y\right)(T-t)\right). \end{aligned} \tag{9}$$

There are then two ways to see that the firm price process exhibits constant volatility. The first notes that the current book value $f$ enters linearly into the price equation (9), while the other terms are time-dependent scaling constants; therefore, volatility is given by that of the firm book price process, equation (3). The second way calculates the elasticity $\epsilon_F$ of the firm price w.r.t. aggregate endowment based on equation (9) and finds $\epsilon_S = 0$ for all $t, f$; then, equation (6) implies that volatility is not stochastic, that it is state-independent and also time-invariant: $V_S = \sigma_F^2$ for all $t, y$. Overall, we conclude both ways that the usual assumption of a representative agent with CRRA preferences leads to constant volatility.

Let us now consider the case where the representative agent does *not* have CRRA preferences. *For illustration*, let us assume for a moment that the state-price function is linear in the endowment $y$, i.e. utility function of the representative agent is equal to quadratic

[8] Additional restrictions need to be imposed that we refrain from working out here.

utility[9], i.e. for a suitable constants $a > 0, \lambda$ we have for all $y > 0$:

$$m(y) = u_r'(y) = a - y, \text{ or equivalently } u_r(y) = -\frac{1}{2\lambda}(a-y)^2 + const. \tag{10}$$

We calculate then $u_r''(y) = a$, such that we find based on equation (4) that

$$\begin{aligned} S(t,f,y) &= f\frac{\gamma(T)}{\gamma(t)}\frac{aE\left[Z_{tT}^F\right] - yE\left[yZ_{tT}^Y Z_{tT}^F\right]}{a-y} \\ &= f\frac{\gamma(T)}{\gamma(t)}\exp\left\{\left(\mu_F - \frac{\sigma_F^2}{2}\right)(T-t)\right\}\frac{a - y\exp\left\{\left(\mu_Y - \frac{\sigma_Y^2}{2} + \kappa\sigma_F\sigma_Y\right)(T-t)\right\}}{a-y}. \end{aligned}$$

This shows that the stock price depends non-linearly on the current aggregate endowment $y = Y_t$; in particular the curvature of the $y$ functional dependence in $S$ is not vanishing. Therefore, we expect volatility to depend on the current endowment. To see this impact more clearly, we calculate the elasticity

$$\epsilon_S(t,y) = -\frac{ay\left(\exp\left\{\left(\mu_Y - \frac{\sigma_Y^2}{2} + \kappa\sigma_F\sigma_Y\right)(T-t)\right\} - 1\right)}{(a-y)\left(a - y\exp\left\{\left(\mu_Y - \frac{\sigma_Y^2}{2} + \kappa\sigma_F\sigma_Y\right)(T-t)\right\}\right)}. \tag{11}$$

This function is not constant in $y$; therefore, in equation (6) the volatility $V_S$ depends on $y$ and in accordance with Theorem 4, we say that volatility is stochastic.

Getting back to the general case, the appendix shows that

$$\epsilon_S(t,y) = (\mu_Y - \kappa\sigma_F\sigma_Y)(T-t)\cdot\left(y\frac{\partial\bar{\rho}}{\partial y}\right)\left(e^{\mu_Y(T-t)}y\right), \tag{12}$$

up to terms of order higher than quadratic in $\sigma_Y$. For *illustration*, let us assume for a moment that this description is accurate. Two insights can be gained from it. *First*, we note that properties of the elasticity $\epsilon_S$ and thereby ultimately the process structure of stochastic volatility are determined by the function $y \mapsto y\frac{\partial\bar{\rho}}{\partial y}$, i.e. essentially the stochastic volatility process is characterized by the first derivative of the representative agent's risk aversion function.

Second, this has implications on the link between stochastic volatility and the functional form of the SPF (the representative agent's risk aversion function). To see this, we recall from

[9]Major disadvantage of the quadratic utility function are (1) that it is decreasing for $y > a$; one may assume $a$ sufficiently large and focus on $0 < y < a$ to address this; (2) that the risk aversion function is increasing for $y < a$ although the literature usually focuses on decreasing risk aversion functions. Despite these shortcomings we use this utility function for a moment, purely for illustrative purposes.

the previous subsection that CRRA preferences for the representative agent is a sufficient condition for constant volatility, while our example of quadratic preferences suggests that this may be a necessary condition, too. Equation 12 suggests that volatility will be stochastic, whenever the function $\frac{\partial \bar{\rho}}{\partial y}$ is not constant (zero); this in turn holds only for CRRA preferences. The appendix shows that it is a necessary condition, indeed. Therefore, we conclude:

**Theorem 4** *Firm price volatility is not stochastic if and only if the representative agent has constant relative risk aversion preferences, respectively, if and only if the SPF is a power function.*

The early option pricing literature provided successful motivations for the Black-Scholes call option pricing formula in single-period setups: Merton (1973) showed that it holds for call options on aggregate endowment in a single-period setup where the representative agent has CRRA preferences and the terminal endowment is lognormal distributed, and Rubinstein (1976) extended this affirmative result to call options on firm value when the terminal firm and endowment claim are bivariate lognormal random variables. Later, Brennan (1979) revisited the original framework in Merton (1973) that considered call options on the endowment and showed that the CRRA preference structure is not only sufficient but that it is also necessary for the Black-Scholes formula to hold. Since then, the option pricing literature has focused on continuous-time trading (hedging) and pricing models of options; it has also come up with elaborate descriptions of asset price processes in so-called stochastic volatility models that describe volatility through an additional process. The derived call option prices from no-arbitrage arguments support the so-called implied volatility smile feature observed in price of traded options. Benninga and Mayshar (2000) linked this back to preferences of investors; interestingly, in an extension of Merton (1973) to an economy where agents have heterogeneous CRRA preferences, they showed that the Black-Scholes formula is no longer valid and reproduce qualitatively the empirical observed implied volatility smile. Theorem 4 ties together these different strands of the literature.

# 4 Economic Foundations

The previous section characterized the functional form of the SPF (risk aversion function of the representative agent) and linked it to stochastic volatility; however, it did not endogenize the risk aversion function. To close this gap, this section extends the setup of the previous section to a pure exchange economy populated by two agents with power utility. This allows us to link the power function property to homogeneity in agent's risk preferences.

## 4.1 The Economy

We study a continuous-time pure exchange economy populated by two agents $i = 1, 2$. Both agents receive at all times $0 \leq t \leq T$ an equal share of the aggregate endowment stream[10], i.e. each agent receives at all times the endowment stream $\frac{Y_t}{2}$.

We say a consumption process is *budget-feasible* if it can be financed by selling the agent's endowment stream and purchasing the consumption process. Because the market is (dynamically) complete, agent $i = 1, 2$ can implement any strictly positive consumption stream $c_i = (c_{it})_t$ that is budget-feasible; she chooses the one that maximizes her time-separable utility

$$E\left[\int_0^T e^{-\gamma t} u_i(c_{it}) dt\right]. \tag{13}$$

Here, $\gamma > 0$ denotes the time-preference parameter. We adopt the concept of a rational expectations equilibrium:

**Definition 5** *An equilibrium consists of consumption processes* $(c_{it})_{0 \leq t \leq T}$ *for both agents* $i = 1, 2$ *which maximize their utility s.t. the budget feasibility condition, and clear the market at all times in all states, i.e. for all* $0 \leq t \leq T$*:* $c_{1t} + c_{2t} = Y_t$*, P-a.s.*

To focus in this paper on the impact of heterogeneity in risk-preferences, we assume both agents have identical *time-preference* parameters. Throughout the remainder of this paper we assume that both agents have preferences with constant relative risk-aversion $\rho_i > 0$ (CRRA preferences, a.k.a. power utility), i.e.

$$u_i(c) = \frac{c^{1-\rho_i}}{1-\rho_i} \text{ for } \rho_i \neq 1 \text{ and } u_i(c) = \ln c \text{ for } \rho_i = 1.$$

[10]This assumption simplifies the exposition but does not affect our results qualitatively.

We will compare an economy where agents are homogeneous in their risk-preferences ($\rho_1 = \rho_2$) with an economy where their risk-preferences are heterogeneous ($\rho_1 \neq \rho_2$).

## 4.2 The Representative Agent's Relative Risk Aversion Function

It is well known, see, e.g. Duffie (2001) that a positive constant $\lambda$ exists with the property that for all $0 \leq t \leq T$, $P$-a.s.,

$$\lambda = \frac{u_2'(c_{2t})}{u_1'(c_{1t})}. \tag{14}$$

**Theorem 6** *For agents with identical time-preferences, firm price volatility is stochastic if and only if agents have heterogeneous risk preferences*

For future reference we denote by $\omega$ the *(market) clearing function*

$$\omega(c) = \lambda^{-1/\rho_2} \cdot c^{\rho_1/\rho_2} + c \tag{15}$$

on the positive real line. Whatever agents' risk-preferences, the function $\omega$ is a strictly increasing, infinitely often differentiable function defined on the positive real line that maps into the positive real line with $\omega(0) = 0$ and $\lim_{x\to\infty} \omega(x) = \infty$. Therefore, it has a unique, infinitely often differentiable, inverse $\psi$ on the positive real line which we call the *sharing rule* of the aggregate endowment. Equation (14) implies $c_{2t} = \lambda^{-1/\rho_2} \cdot c_{1t}^{\rho_1/\rho_2}$; the market clearing condition then reads for all $0 \leq t \leq T$:

$$Y_t = c_{1t} + c_{2t} = c_{1t} + \lambda^{-1/\rho_2} \cdot c_{1t}^{\rho_1/\rho_2} = \omega(c_{1t}), \text{ i.e. } c_{1t} = \psi(Y_t) \ P\text{-a.s.} \tag{16}$$

In complete markets, the marginal utility of either representative agent defines the state-price process as $M_t = \gamma(t) \cdot m(Y_t)$, where the $\gamma, m$ are functions defined by

$$\gamma(t) = \exp(-\gamma \cdot t) \text{ and } m(y) = \psi^{-\rho_1}(y) \tag{17}$$

for $0 \leq t \leq T, y > 0$. Note that here $m$ defines the state-price density[11] in the sense of definition 1. Based on this we find the relative risk aversion function of the representative agent as

$$\bar{\rho}(y) = -\frac{u_r''(y)y}{u_r'(y)} = -\frac{m'(y)y}{m(y)} = \rho_1 \frac{y}{\psi}. \tag{18}$$

[11]It is tedious to prove that suitable differentiability properties hold; we refrain from presenting this here.

A closed-form sharing rule is available for quadratic market-clearing functions[12], i.e. when agents are heterogeneous with a ratio of relative risk aversion parameters $\rho_1/\rho_2 = 2$ or equivalently $\rho_1 = 2\rho_2$. *For illustration*, let us adopt this parameterization for a moment; then, for given $\rho_2$, equation (15) has a unique inverse

$$\psi(y) = \frac{1}{2}\left(\lambda^{\frac{1}{2\rho_2}}\sqrt{\lambda^{\frac{1}{\rho_2}} + 4y} - \lambda^{\frac{1}{\rho_2}}\right), \text{ such that } m(y) = 2^{2\rho_2}\left(\lambda^{\frac{1}{2\rho_2}}\sqrt{\lambda^{\frac{1}{\rho_2}} + 4y} - \lambda^{\frac{1}{\rho_2}}\right)^{-2\rho_2}; \tag{19}$$

and so

$$\bar{\rho}(y) = \frac{4\rho_2 y}{4y + \lambda^{\frac{1}{\rho_2}} - \lambda^{\frac{1}{2\rho_2}}\sqrt{4y + \lambda^{\frac{1}{\rho_2}}}}. \tag{20}$$

Clearly the function is a weighted average of individual agent's relative risk aversion, i.e. the representative agent's relative risk aversion function is *not constant*. This result suggests that we take a closer look at the functional form the the representative agent's risk aversion function with homogeneous/heterogeneous agents.

## 4.3 Heterogeneity and Stochastic Volatility

We first look at the case of homogeneous agents ($\rho_1 = \rho_2 = \rho$). Then, equations (15, 17) read

$$\omega(c) = (1 + \lambda^{-1/\rho}) \cdot c, \psi(y) = \frac{1}{1 + \lambda^{-1/\rho}} y, \text{ and } m(y) = (1 + \lambda^{-1/\rho})^{\rho} y^{-\rho}.$$

Under this assumption, the sharing rule $\psi$ is linear, a well-known result for homogeneous CRRA preferences, see, e.g. Magill and Quinzii (1996). We then calculate $\bar{\rho} = const$, again a well known result, see, e.g., Rubinstein (1976). Together with theorem 4 this shows that volatility is not stochastic with homogeneous agents.

Finally, we look at the heterogeneous agent case ($\rho_1 \neq \rho_2$). When $\rho_1, \rho_2$ take any positive value, there are no closed-form solutions for the sharing rule, in general. However, it is well known that at the extremes $y \downarrow 0$ and $y \to \infty$ different agents dominate the state price function, i.e. asymptotically it is $y^{-\rho_1}$ at one and like $y^{-\rho_2}$ at the other extreme. However, ultimately, it cannot be described by a single power function of type $y^{-\bar{\rho}}$.

[12] Wang (1996) noted this and pointed out that closed-form solutions are available also for market clearing functions that are third and for fourth order polynomials, i.e. $\rho_1/\rho_2 = 3$ or 4. He looked at the link between the aggregate consumption stream and short-term interest rates, but was *not* interested in the volatility dynamics.

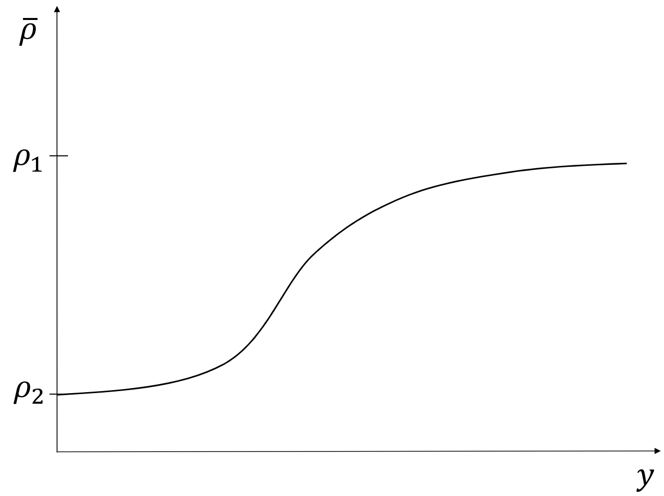


Figure 1: The representative agent's risk aversion function; $\rho_2 < \rho_1$.

Figure 1 illustrates the representative agent's relative risk aversion function $\bar{\rho}$; without loss of generality we order here the agents such that $\rho_2 < \rho_1$. For $y \downarrow 0$, the second agent dominates such that the $\bar{\rho} \to \rho_2$. However, for $y \to \infty$, the first agent dominates such that $\bar{\rho} \to \rho_1$. Therefore, the representative agent's relative risk aversion function $\bar{\rho}$ is *not* constant in $y$ for any (heterogeneous) CRRA agents ($\rho_1 \neq \rho_2$). Based on these observations, the representative agent's relative risk aversion function cannot be constant and we see stochastic volatility according to theorem 4. The appendix proves:

**Theorem 7** *For CRRA agents with identical time-preference parameters, firm price volatility is stochastic if and only if agents have heterogeneous risk-preferences.*

This is our main result in this paper. It relates stochastic property of the volatility process to heterogeneity in agent's risk preferences: heterogeneity in risk preferences leads to stochastic volatility, while homogeneity leads to constant volatility. Our results tie together insights from the single-period literature with those from the continuous-time pricing literature. To recall how, a first observation is based on the discrete-time (single-period) literature: that strand of literature studied that options written on lognormal distributed aggregate endowment and documented for CRRA agents that homogeneity in risk preferences is a necessary and sufficient condition for the Black-Scholes formula to hold (Merton (1973), Brennan (1979), Benninga and Mayshar (2000)); this stochastic framework cannot study dynamic aspects. A second observation is based on the continuous-time asset pricing literature: we note that a (univariate) geometric Brownian endowment process is the continuous-time analogue of a lognormal distributed endowment at terminal date; Weinbaum (2009) found for such a continuous-time setup that volatility is a function of the endowment (so-called state dependent volatility) when agents have heterogeneous CRRA risk preferences; while this is an important insight, the literature documented extensively that volatility in asset price processes is a stochastic process itself. Finally, a third observation is that the stochastic volatility continuous-time option pricing literature matches well the empirical departure of call prices from the Black-Scholes formula. Whereas the literature focuses largely on endowment claims, a major modeling component in this paper is to revisit Rubinstein (1976) and extend it to continuous-time, as he considers expressively claims on firm value. The single-period distributions can be interpreted as the distributions of a continuous-time process; with this in mind our characterization of (non-) stochastic volatility unifies the three before-mentioned observations.

# 5 Conclusion

This paper linked the functional form of stochastic discount factor to stochastic volatility. In particular, we showed that we see to constant volatility only when the representative agent exhibits CRRA preferences, while non-CRRA preferences lead to stochastic volatility of the firm price process. We derived the equilibrium in an economy populated by two agents with identical time-preference parameters but potentially heterogeneous CRRA preferences and showed that the firm price process exhibits stochastic volatility if and only if agents' risk preferences are heterogeneous. Our results unified several theoretical observations from the single-period and continuous time option pricing literature as well as those from the empirical literature on observed price processes and traded option prices. Different from the stochastic dividends, stochastic information flow and heterogeneous beliefs literature, we provide an alternative rationale for stochastic volatility based on aggregation of heterogeneous agents.

# Appendix: Proofs

Throughout this appendix we study state-price processes in the sense of definition 1 and use the random variables $Z^Y_{tT}, Z^F_{tT}$ of equation (2.2). For later reference we denote by $\epsilon_m$ the elasticity of the state-price function $m$ w.r.t. the aggregate endowment:

$$\epsilon_m(y) \;=\; \frac{m'(y)y}{m(y)}.$$

Note that the elasticity $\epsilon_m$ relates to the relative risk aversion function $\bar{\rho}$ of the representative agent through

$$\bar{\rho}(y) = -\frac{u_r''(y)y}{u_r'(y)} = -\frac{m'(y)y}{m(y)} = -\epsilon_m(y). \tag{A1}$$

Also for later references, we define a time functions $\varphi, \tilde{\varphi}$ and denote by $\epsilon_B$ the elasticity of the function $y \mapsto E[m(yZ^Y_{tT})]$ w.r.t. the aggregate endowment:

$$\varphi(t) \;=\; e^{\mu_Y(T-t)}, \tilde{\varphi}(t) = e^{\kappa\sigma_Y\sigma_F(T-t)}; \epsilon_B(t,y) = \frac{E\left[m'(yZ^Y_{tT})\cdot(yZ^Y_{tT})\right]}{E[m(yZ^Y_{tT})]}.$$

**Lemma A1** *For a state-price function $m$, $y > 0$ and $0 < t < T$ we have*

$$E\left[m(yZ^Y_{tT})\cdot Z^F_{tT}\right] \;=\; e^{\mu_F(T-t)}E[m(\tilde{\varphi}yZ^Y_{tT})].$$

**Proof of Lemma A1.** The random variable $e^{-\mu_F(T-t)}Z^F_{tT}$ fulfills the properties of a Radon-Nikodyn density and can be used to define a new probability measure $Q^F$. Under this new probability measure, the random variable $\ln Z^Y_{tT}$ has mean $(\mu_Y - \sigma_Y^2/2)(T-t) + \kappa\sigma_Y\sigma_F(T-t)$. Thus, we can treat $Z^Y_{tT}$ as $\tilde{\varphi}Z^Y_{tT}$ under the original measure which implies the statement. ■

**Proof of Theorem 2.** We define the process $\tilde{W}^F$ and the function $V_S$ by setting

$$d\tilde{W}^F_t \;=\; \frac{1}{\sqrt{V_S}\cdot S_t}\left(\frac{\partial S}{\partial f}\sigma_F F_t dW^F_t + \frac{\partial S}{\partial y}\sigma_Y Y_t dW^Y_t\right) \text{ and} \tag{A2}$$

$$V_S(t,f,y) \;=\; \frac{1}{S^2}\left\{\left(\frac{\partial S}{\partial f}\sigma_F f\right)^2 + 2\kappa\frac{\partial S}{\partial f}\sigma_F f\frac{\partial S}{\partial y}\sigma_Y y + \left(\frac{\partial S}{\partial y}\sigma_Y y\right)^2\right\}. \tag{A3}$$

It is straightforward to check that the process $\tilde{W}^F$ has independent increments as well as that these are conditionally normal distributed with a mean of zero and a variance equal to the time increment; therefore, $\tilde{W}^F$ describes a standard Wiener process. Based on equation

(4) we calculate $\frac{\partial S}{\partial f} = \frac{S}{f}$; using the definition of the elasticity $\epsilon_S$, see equation (5) proves the statement with

$$V_S(t,y) = \sigma_F^2 + \big(\sigma_Y \epsilon_S(t,y)\big)^2 + 2\kappa\sigma_F\sigma_Y\epsilon_S(t,y).$$

The stated form of $V_S$ follows through a straightforward transformation. ■

**Proposition A2** *At any point in time, we can write*

$$\epsilon_S(t,y) = \epsilon_m(\varphi y) - \epsilon_m(y) + \kappa\sigma_F\sigma_Y(T-t)\cdot\left(y\frac{\partial \epsilon_m}{\partial y}\right)\left(e^{\mu_Y(T-t)}y\right),$$

*up to terms of order higher than quadratic in $\sigma_Y$.*

**Proof of Proposition A2.** To simplify presentation in this proof we do not write out the time-dependency of $\tilde{\varphi}$ explicitly, unless necessary to prevent confusion. Lemma A1 proves that the firm price function of equation (4) can be written as

$$S(t,f,y) = e^{\mu_F(T-t)} f \frac{\gamma(T)}{\gamma(t)} \frac{E[m(y\tilde{\varphi}Z_{tT}^Y)]}{m(y)}.$$

Based on this representation we calculate

$$\frac{\partial S}{\partial y} = e^{\mu_F(T-t)} f \frac{\gamma(T)}{\gamma(t)} \frac{E\left[m'(y\tilde{\varphi}Z_{tT}^Y)\tilde{\varphi}Z_{tT}^Y\right]m(y) - E\left[m(y\tilde{\varphi}Z_{tT}^Y)\right]m'(y)}{m^2(y)}.$$

Using the above characterization of $S$ we find

$$\epsilon_S(t,y) = \epsilon_B(t,\tilde{\varphi}y) - \epsilon_m(y) \tag{A4}$$

For all $\sigma_Y > 0$ and $y > 0$ we define a random variable

$$\phi(\sigma_Y, y) = y\exp\left(\left(\mu_Y + \kappa\sigma_F\sigma_Y - \frac{\sigma_Y^2}{2}\right)(T-t) + \sigma_Y\sqrt{T-t}U\right),$$

where $U$ is a given standard normal random variable. To simplify the presentation in this proof we do not write out the dependencies of $\phi$ on $\sigma, y$ explicitly, unless necessary to prevent confusion. We then define functions $\xi_1, \xi_2$ and $\zeta_1, \zeta_2$ by

$$\begin{aligned} \xi_1(\sigma_Y, y) &= m'(\phi)\,\phi, \xi_2(\sigma_Y, y) = m(\phi), \\ \text{and } \zeta_1(\sigma_Y, y) &= E[\xi_1(\sigma_Y, y)], \zeta_2(\sigma_Y, y) = E[\xi_2(\sigma_Y, y)]. \end{aligned}$$

Our idea is to study the asymptotic behavior of $\epsilon_B(t, \tilde{\varphi}y)$ in the parameter $\sigma_Y$ near 0. Note that $y\tilde{\varphi}Z_{tT}^Y = \phi$, such that $\epsilon_B(t, \tilde{\varphi}y) = \zeta_1(\sigma_Y, y)/\zeta_2(\sigma_Y, y)$. Under suitable assumptions on the state-price function we can interchange first and second order $\sigma_Y$ differentiation in $\zeta_1$, $\zeta_2$ and expectation. Furthermore, we can apply Taylor's Theorem; it tells us that

$$\zeta_1(\sigma_Y, y) = \zeta_1(0, y) + \frac{\partial \zeta_1}{\partial \sigma_Y}(0, y)\sigma_Y, \quad \zeta_2(\sigma_Y, y) = \zeta_2(0, y) + \frac{\partial \zeta_2}{\partial \sigma_Y}(0, y)\sigma_Y,$$

both up to terms of order higher than quadratic in $\sigma_Y$. This expansion implies

$$\frac{\zeta_1(\sigma_Y, y)}{\zeta_2(\sigma_Y, y)} = \frac{\zeta_1(0, y)}{\zeta_2(0, y)} + \left( \frac{\frac{\partial \zeta_1}{\partial \sigma_Y}(0, y)}{\zeta_2(0, y)} - \frac{\zeta_1(0, y)\frac{\partial \zeta_2}{\partial \sigma_Y}(0, y)}{\zeta_2^2(0, y)} \right) \sigma_Y, \tag{A5}$$

up to terms of order higher than quadratic in $\sigma_Y$. For further analysis we then calculate

$$\frac{\partial \zeta_1}{\partial \sigma_Y} = E\left[ m''(\phi)\,\phi \frac{\partial \phi}{\partial \sigma_Y} + m'(\phi)\frac{\partial \phi}{\partial \sigma_Y} \right], \text{ and } \frac{\partial \zeta_2}{\partial \sigma_Y} = E\left[ m'(\phi)\frac{\partial \phi}{\partial \sigma_Y} \right].$$

In addition, we calculate

$$\begin{aligned} \frac{\partial \phi}{\partial \sigma_Y} &= \phi \cdot \left( \kappa\sigma_F(T-t) - \sigma_Y(T-t) + \sqrt{T-t}U \right), \text{ such that} \\ \phi(\sigma_Y = 0) &= \varphi y, \frac{\partial \phi}{\partial \sigma_Y}(\sigma_Y = 0) = \varphi y \cdot \left( \kappa\sigma_F(T-t) + \sqrt{T-t}U \right). \end{aligned}$$

At $\sigma_Y = 0$ we then find that

$$\begin{aligned} \zeta_1(0, y) &= m'(\varphi y)\,\varphi y, \frac{\partial \zeta_1}{\partial \sigma_Y}(0, y) = \left( m''(\varphi y)\, y\varphi + m'(\varphi y) \right) y\varphi\kappa\sigma_F(T-t), \\ \zeta_2(0, y) &= m(\varphi y)\,, \frac{\partial \zeta_2}{\partial \sigma_Y}(0, y) = m'(\varphi y)\,\varphi y\kappa\sigma_F(T-t). \end{aligned}$$

Using equation (A5) this shows

$$\epsilon_B(t, \tilde{\varphi}y) = \frac{m'(\varphi y)\,\varphi(t) y}{m(\varphi y)} + \left( \frac{m''(\varphi y)\,\varphi y + m'(\varphi y)}{m(\varphi y)} - \frac{\left(m'(\varphi y)\right)^2 \varphi y}{m^2(\varphi y)} \right) y\varphi\kappa\sigma_F\sigma_Y(T-t),$$

In addition, we calculate that

$$\begin{aligned} &\frac{m''(\varphi y)\,\varphi y + m'(\varphi y)}{m(\varphi y)} - \frac{\left(m'(\varphi y)\right)^2 \varphi y}{m^2(\varphi y)} = \frac{\partial}{\partial y}\left( \frac{m'(\varphi y)\, y}{m(\varphi y)} \right) = \frac{1}{\varphi}\frac{\partial}{\partial y}\left( \frac{m'(\varphi y)\, y\varphi}{m(\varphi y)} \right) \\ = \quad & \frac{1}{\varphi}\frac{\partial \epsilon_m(\varphi y)}{\partial y} = \frac{\partial \epsilon_m}{\partial y}(\varphi y)\,. \end{aligned} \tag{A6}$$

Overall, this shows

$$\epsilon_B(t, \tilde{\varphi}y) = \epsilon_m(\varphi y) + \left( y\frac{\partial \epsilon_m}{\partial y} \right)(\varphi y)\,\kappa\sigma_F\sigma_Y(T-t),$$

Using equation (A4) then ends the proof. ■

**Proposition A3** *At any point in time, we can write*

$$\epsilon_S(t,y) = (\mu_Y - \kappa\sigma_F\sigma_Y)(T-t)\cdot\left(y\frac{\partial\bar{\rho}}{\partial y}\right)\left(e^{\mu_Y(T-t)}y\right),$$

*up to terms of order higher than quadratic in $\sigma_Y$.*

**Proof of Proposition A3.** Proposition A2 together with the equality (A1). ■

**Proof of Theorem 4.** Based on our discussion before Theorem 4 it remains to prove the converse. Therefore, in this proof let us assume that firm volatility is not stochastic; our goal is to show that this implies that $m$ is a power function. As discussed earlier, this will then imply that the representative agent has CRRA preferences.

Our idea is to study the asymptotic behavior of $\epsilon_S$ in the parameter $t$ near $T$; then drift and volatility tend to zero. A proof using a time expansion that is analogous to the $\sigma_Y$ expansion of Proposition A2 shows that

$$\epsilon_S(t,y) = \epsilon_m(\varphi y) - \epsilon_m(y) + \kappa\sigma_F\sigma_Y(T-t)\cdot\left(y\frac{\partial\epsilon_m}{\partial y}\right)\left(e^{\mu_Y(T-t)}y\right),$$

up to terms of order higher than quadratic in $T-t$. A Taylor expansion of $\epsilon_m$ around $\varphi y$ shows that $\epsilon_m(\varphi y) - \epsilon_m(y) = \frac{\partial\epsilon_m}{\partial y}(\varphi y)(1-\varphi)y$, up to terms of order higher than quadratic in $1-\varphi$. Also, a Taylor expansion shows that $\varphi = 1+\mu_Y(T-t)$, up to terms of order higher than quadratic in $T-t$. Overall, this proves

$$\epsilon_S(t,y) = \left(y\frac{\partial\bar{\rho}}{\partial y}\right)(\varphi y)\left(\mu_Y - \kappa\sigma_F\sigma_Y\right)(T-t),$$

up to terms of order higher than quadratic in $T-t$. This implies that $\frac{\partial\bar{\rho}}{\partial y}y$ must vanish for all $y$, i.e. $\bar{\rho}$ must be a constant. ■

**Proof of Theorem 7.** We only prove that the converse is true, i.e. we prove that if volatility is not stochastic, then both agents have identical risk-aversion parameters. The result follows directly from this.

For this, we assume that volatility is not stochastic. Theorem 4 implies that there are constants $\alpha, \beta$ such that the state-price function is $m(y) = \alpha y^\beta$. Also, we know that $m(y) = (\psi(y))^{-\rho_1}$. Therefore,

$$\alpha^{-1/\rho_1}y^{-\beta/\rho_1} = \psi(y), \text{ and } \lambda^{-1/\rho_2}\alpha^{-1/\rho_2}y^{-\beta/\rho_2} = \lambda^{-1/\rho_2}(\psi(y))^{\rho_1/\rho_2},$$

which implies

$$y = \omega(\psi(y)) = \psi(y) + \lambda^{-1/\rho_2}(\psi(y))^{\rho_1/\rho_2} = \alpha^{-1/\rho_1} y^{-\beta/\rho_1} + \lambda^{-1/\rho_2}\alpha^{-1/\rho_2} y^{-\beta/\rho_2}.$$

However this requires $\rho_1 = \rho_2$, i.e. agents are homogeneous. ■

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