Dynamic loss modeling
for heterogeneous credit portfolios∗

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July 2, 2015

Abstract

Extant intensity based models for credit portfolio losses with heterogeneous default rates and heterogeneous exposure sizes have an important shortcoming viz. they usually require computationally expensive Monte Carlo simulations. CreditRisk+ is a notable exception that allows for computation of the loss distribution analytically. With moderate restrictions on dependency modelling, it uses generating functions to compute the loss probabilities quickly and accurately. However, this advantage is overshadowed by the fact that it is a static single period model. This is a major drawback when working with portfolio exposures having different maturities and when pricing instruments where the term structure of default rates matters. The framework proposed here incorporates time varying default rates and volatilities that may differ across names as well. Equally important is the fact that this framework does not require exposure banding (as CreditRisk+ does). The evolution of loss distribution with time can be modelled using the Cox-Ingersoll-Ross processes as latent macroeconomic processes driving the dynamic default intensities. The characteristic function of the credit portfolio loss can be obtained explicitly. Using the Fast Fourier Transform it can be inverted to obtain the portfolio loss distribution in a numerically stable manner.

∗The authors are grateful for helpful feedback from seminar participants at the Conference on Advanced Mathematical Methods for Finance, Vienna University of Technology.
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1 Introduction

Extant models for portfolio losses with heterogeneous default rates and heterogeneous exposure sizes have an important shortcoming viz. they require computationally expensive Monte Carlo simulations. CreditRisk+ is a notable exception that allows for computation of the loss distribution analytically. It imposes moderate restrictions on dependency modelling, and using the generating functions technique to compute the loss probabilities quickly and accurately. However, this advantage is overshadowed by the fact that it is a static single period model.

The static nature of the CreditRisk+ framework is a major drawback when measuring credit risk of portfolios with exposures having different maturities. The problem is even more pronounced if the model is used in the context of pricing credit derivative instruments where the term structure of default rates matters. The framework proposed here incorporates time varying default rates and volatilities. Furthermore, it allows for these default rates and volatilities (as well as exposure amounts) to vary across names.
Additionally, an important feature of this framework is that it does not require exposure banding (as CreditRisk+ does). It is possible to relax the requirement for loss discretization by computing the characteristic function of the credit portfolio loss, as against the probability generating function (which is what CreditRisk+ uses). Using the Fast Fourier Transform the characteristic function can be inverted to obtain the portfolio loss distribution in a numerically stable manner.

The dynamic nature of the default process is modeled via integrated Cox-Ingersoll-Ross (henceforth CIR) processes as latent macroeconomic processes driving the dynamic default intensities. Integrated CIR processes are used to scale the one period default probability up or down as time evolves. The characteristic function of the integrated CIR process can be computed explicitly; this is what facilitates the computation of the portfolio loss distribution.

2 Literature Review

Traditionally, credit risk models broadly fell into two categories viz. structural vs. reduced form models. Structural models attempt to model the obligor firm’s fundamentals. The seminal work in this area
was done by Black and Scholes 1973 and Merton 1974. The main idea was that a default occurs at maturity if debt issuer’s assets valued on maturity date are less than the face value of maturing debt, leading to valuation of debt using option pricing theory. The value of a risky zero-coupon bond issued by the obligor firm in this case can be expressed as the value of equivalent riskless zero-coupon bond less Black-Scholes-Merton value of a European put option on the firm’s assets. Subsequent modifications of this approach introduced more flexibility in this basic construct such as: default prior to maturity, coupons prior to maturity, stochastic interest rates, deviation from strict absolute priority rule, presence of taxes and bankruptcy costs etc. See for instance Black and Cox 1976, Geske 1977, Leland 1994, Longstaff and Schwartz 1995 among others.

While structural models were firmly grounded in economic theory, making transparent the relationship between a firm’s capital structure and its credit risk, their (endogenous) default predictions were inconsistent with empirical observations. They under-predicted credit spreads and default probabilities for short horizon low risk debt (see for instance P. Jones and Rosenfeld 1984 and Franks and Torous 1989). Further, it became clear that distance to default, a volatility
adjusted measure of leverage, popularised by structural models, did not entirely explain variation in default probabilities across firms (see for instance Bharath and Shumway 2008). These shortcomings likely fueled the advance of credit risk literature in reduced-form models which treat the firm’s (exogenous) default as a surprise event.

Surprise defaults allowed better fit with observed credit spreads. Early reduced form models such as Litterman and Iben 1991, R. Jarrow and Turnbull 1997, Madan and Unal 1998 and Duffie and Singleton 1999 soon spawned off a vast family of models whose main object of interest was the firm’s default intensity i.e. the firm’s instantaneous propensity to default, conditional on survival to date (see Lando 2004 and Bielecki and Rutkowski 2004 for a comprehensive overview of intensity based models).

An important category of reduced form models are rating migration models where the firm’s credit riskiness is discretized to a set of credit ratings. The worst rating is mapped to default state. Given a time horizon and some initial rating, the migration model can compute the probability of migration to all ratings, and hence to default state as a special case. The stochastic process governing the dynamics of rating
migrations is presumed to be a Markov chain (e.g. R. Jarrow and Turnbull 1997). However, there is also sufficient evidence in literature that the current rating of a firm does not capture the effect of business cycle on default rates (see for instance P. Nickell and Varotto 2000) nor the effect of prior ratings history (see for instance Lando and Skodeberg 2002). The subject of heterogeneity in the rating migration probabilities of different firms of the same current rating (in a continuous time Markov chain setting) is explored in detail by Kadam and Lenk 2008.

As an alternative to Markov chains, doubly stochastic processes are another popular way of modeling defaults. In this model family, firms defaults are correlated only as implied by the correlation of factors determining their default intensities. Although S. Das and Saita 2007 provided empirical evidence against the standard doubly stochastic model assumption, D. Duffie and Wang 2007 demonstrated quite a good empirical fit (out of sample, as measured by accuracy ratios) for a model of this type. However, in practice their model can turn out to be difficult to implement when different forecast horizons are required, and time series inputs that determine the evolution of the intensity process make the dimensionality of the state space very high.
J. Duan and Wang 2012 provided an improved version, and made the specification tractable by employing a forward intensity model in which default probabilities for different horizons are computed based only on the input variables at the time the prediction is made.

Having modeled and estimated an individual obligor’s probability of default, the next practical requirement for a financial institution exposed to credit risk is to compute the probability distribution of losses from a collection of default risky obligors (e.g. a basket of corporate bonds, a loans portfolio etc). There are three industry standards for this. First, based on the credit migration approach, as proposed by JP Morgan with CreditMetrics, described in J. P. Morgan 1997. Second, the option pricing, or structural approach, as initiated by KMV, based on the Merton model described in Crosbie and Bohn 2003. Third, the actuarial approach as proposed by Credit Suisse Financial Products (CSFP) with CreditRisk+, described in CreditSuisse 1997, which focuses only on default and which assumes that default of individual obligors follows an exogenous Poisson process. An excellent overview of all three, as well as a comparative analysis can be seen in M. Crouhy and Mark 2000. In a similar vein, Wieczerkowski 2004 explores the relationship between CreditRisk+ and CreditMetrics by showing that
there exists in general a consistent parametrization of that results in
the same loss distribution.

This paper builds on ideas from the CreditRisk+ framework. While
analytical tractability of CreditRisk+ made it a popular approach, it
also motivated several extensions and enhancements, many of which
can be found in Gundlach and Lehrbass 2004. This paper can be con-
sidered as one such enhancement, extending the notion of risk factors
from static to dynamic.

3 Model

3.1 Static Version

3.1.1 Similarities with CreditRisk+

Consider a portfolio of credits with \( m \) obligors. Denote default
event of obligor \( A \) by \( \mathbbm{1}_A \) and the corresponding non-negative loss
given default by \( \nu_A \in \mathbb{R}^+ \). Thus the credit portfolio loss is given by
\[ L = \sum_A \mathbbm{1}_A \nu_A. \] Our aim is to assess risk measures such as value at
risk (VaR) or expected shortfall (ES) for this credit portfolio. These
summary statistics can be conveniently obtained from the probability
distribution of the portfolio losses. The static model we present now is similar to the simplest version of CreditRisk+ model as described in its technical document (CreditSuisse 1997).

CreditRisk+ framework yields the loss distribution in semi-explicit form, and computes it efficiently and accurately at the cost of imposing two restrictions. First, the dependency between obligor defaults is modelled entirely by the influence exerted by risk factors with known default rates and volatilities. Second, the state space for loss random variables is discrete and small. The latter is achieved via exposure banding therefore choosing a correct band-size is essential to accuracy and numerical stability.

3.1.2 Sector variables

To remain consistent with the CreditRisk+ documentation we refer to the risk factors as sectors. In theory these sectors could represent any systemic risk factors such as industry sectors or countries of domicile or more complex combinations of exposure characteristics. Each credit is characterized by its default probability $p_A = P(\mathbb{I}_A = 1)$ which depends on its sector weights $\omega_{Al}$ and its loss given default $\nu_A \in \mathbb{N}$ expressed in units of band-size. Gamma distributed mutu-
ally independent $K$ sector variables $S_k \overset{d}{=} \Gamma_{\alpha,\beta} \ k \in \{1, \ldots, K\}$ influence the default probability $p_A$ via sector weights i.e. $p_A^S = p_A|S = p_A \cdot \sum_{k=1}^{K} \omega A_k S_k$. The sector variables are normalized i.e. $ES_k = 1$, $k \in \{1, \ldots, K\}$.

### 3.1.3 Probability Generating Function

For every obligor loss $L_A := \nu_A \cdot \mathbb{1}_A$ and any non-negative $z \in \mathbb{R}^+$ we can define the conditional expectation $G_{L_A}(z|S) = \mathbb{E}(z^{\nu_A} \cdot \mathbb{1}_A|S) = 1 + p_A^S(z^{\nu_A} - 1)$ which is approximately equal to $\exp(p_A^S(z^{\nu_A} - 1))$.

Assuming independence between obligors gives for the total portfolio loss a similar conditional expectation $G_L(z|S) = \prod_A G_{L_A}(z|S)$ which is straightforward to compute. Un-conditioning over the probability distribution of each sector specific risk factor gives the Probability Generating Function (henceforth PGF) of portfolio loss as $G_L(z) = \int G_L(z|S = s)dF_{\alpha,\beta}(s)$. It turns out that this integral can be computed explicitly due to the analytically tractable functional forms assumed for the sector specific risk factors. More importantly, the calculation of the probabilities by taking derivatives at $z$ approaching 0 is vastly simplified using Panjer recursion. The final result is that loss probabilities can be quickly and explicitly computed in a reasonably
3.2 Dynamic Version

3.2.1 From static to dynamic

The aim of this paper is to extend the above framework to incorporate the time dimension. In that sense we are proposing a dynamic version of CreditRisk+. However, as mentioned in the next section, the treatment of exposures can be made more general than in CreditRisk+. Moving from the static setup described above to the dynamic setup proposed involves the following changes:

- Indicator random variable $1_A$ becomes an indicator process $1_A(t)$

- $K$-dimensional vector $S$ of risk factors is replaced by a $K$-dimensional stochastic process $(S(t))_{t \geq 0}$ which generates the filtration $\mathcal{F}_t$

- Additional random variable introduced of interest now is $\tau_A$ the default time of obligor $A$.

- Static one period default probabilities $p_A^S$ are now replaced by time-dependent probabilities i.e. $p_A^S(t) = P(\tau_A \leq t|\mathcal{F}_t)$
3.2.2 Hazard rate

The hazard rate $\lambda(t)$ of a default event is defined as the instantaneous probability of default given that no default has happened before:

$$\lambda(t) := \lim_{\Delta t \to 0} \frac{P(\mathbb{1}(t + \Delta t) = 1|\mathbb{1}(t) = 0)}{\Delta t}$$

(1)

If $F$ is the cdf of the default probability, and the pdf $f$ exists, then we have the relation

$$\lambda(t) = \frac{f(t)}{1 - F(t)},$$

(2)

resulting in

$$P(\mathbb{1}(t) = 1) = F(t) = 1 - e^{\int \lambda(v) dv}.$$  

(3)

3.2.3 Risk factor processes

The natural transition from the static CreditRisk+ model to a dynamic one is to model the instantaneous default probability (i.e. the hazard rate, default rate) being linearly dependent on factor processes. This is done as follows: we characterize the time-varying default rates as a product of a baseline one period default rate $p_{A0}$ and a time vary-
ing adjustment process. The adjustment process is a weighted sum of CIR processes $S_k(t)$; one such process for each sector that the credit belongs to. Hence, the hazard rate of obligor $A$ is determined by

$$\lambda_{A0}(t) = p_{A0} \sum_{k=1}^{K} \omega_{Ak} S_k(t).$$

(4)

Thus for each credit, we require an input for a one-period default probability $p_A = P(\mathbb{1}_A(1) = 1)$ as we did in the static model. In addition we need to define $(S_k(t))_{t \geq 0}$, which are independent CIR processes with parameters $\lambda_k$ (speed), $\eta_k$ (level), $\theta_k$ (square root coefficient)

$$dS_k(t) = \lambda_k(\eta_k - S_k(t))dt + \theta_k \sqrt{S_k(t)}dW_t$$

When using the integrated CIR processes $Y_{k,t} := \int_0^t S_k(v)dv$ to adjust the baseline default probability we also need starting points $y_{k}^0$.

3.2.4 PGF of portfolio loss

**Portfolio loss** We allow for an exposure to belong to multiple sectors. This has been done in CreditRisk+ and is achieved via sector weights $\omega_{Ak}$ that sum to 1. The sector-specific weights define proportion of influence exerted by sectors on an exposure. In the frame-
work we propose, sector weights are used in defining loss probabilities of individual exposures. Thus, they will affect the probability of incurring a particular portfolio loss at time $t$. Denote the loss by $L(t) = \sum_A \mathbb{1}_A(t)\nu_A$ where $\nu_A$ is specified in units of exposure band size i.e. for now, we resort to exposure banding as in the static model so that the loss random variable is defined on a small discrete state space.

**PGF formulation** We seek the PGF of this portfolio loss $G_{L_t}(z) = \mathbb{E}(G_{L_t}(z|S))$. As in the static case we assume obligor defaults are independent conditional on the evolution of the sector specific risk factors. Thus $G_{L_t}(z|S) = \prod_A G_{L_{tA}}(z|S)$ and the portfolio loss PGF becomes

$$G_{L_t}(z) = \mathbb{E}\left(\prod_A G_{L_{tA}}(z|S)\right)$$

$$= \mathbb{E}\left(\prod_A 1 + (z^{\nu_A} - 1)p^S_A(t)\right)$$
using $\forall \delta \approx 0, e^\delta \approx 1 + \delta$ we get

\[
\approx E \left( \prod_A \exp \left( (z^{\nu_A} - 1)p^S_A(t) \right) \right) 
\]

\[
= E \left( \exp \left( \sum_A (z^{\nu_A} - 1)p^S_A(t) \right) \right)
\]

\[
= E \left( \exp \left( \sum_A (z^{\nu_A} - 1) \left[ 1 - e^{-\int_0^t \lambda^S_A(v) \, dv} \right] \right) \right)
\]

again using $\forall \delta \approx 0, e^\delta \approx 1 + \delta$ we get

\[
\approx E \left( \exp \left( \sum_A (z^{\nu_A} - 1) \int_0^t \lambda^S_A(v) \, dv \right) \right) 
\]

\[
= E \left( \exp \left( \sum_A (z^{\nu_A} - 1) \left( p_A \sum_{k=1}^K \omega_{Ak} \int_0^t S_k(v) \, dv \right) \right) \right)
\]

\[
= E \left( \exp \left( \sum_{k=1}^K \sum_A (z^{\nu_A} - 1) p_A \omega_{Ak} \int_0^t S_k(v) \, dv \right) \right)
\]

\[
= E \left( \exp \left( \sum_{k=1}^K \xi_k(z) \cdot Y_{k,t} \right) \right)
\]

where $\xi_k(z) = \sum_A (z^{\nu_A} - 1) \omega_{Ak} p_A$ and $Y_{k,t}$ is the integrated CIR process for sector $k$. Note that the approximations (5) and (6) are acceptable because $p^S_A(t)$ and $\int_0^t S_k(v) \, dv$ are small if $t$ is not too large.

These approximations are equivalent to a Poisson approximation of the default process. For small one-year default probabilities and not too long time horizons (e.g. $t \leq 10$), using Poisson approximation for
probabilities of multiple defaults is not unreasonable. This expectation can be explicitly computed using the fact that the Laplace transform of the integrated CIR process \( Y_{k,t} := \int_0^t S_k(v)dv \) is

\[
\mathbb{E}(e^{-u Y_{k,t}}) = \exp\left(\frac{\lambda_k^2 \eta_k t}{\theta_k^2}\right) \left( \cosh \frac{\gamma_k t}{2} + \frac{\lambda_k}{\gamma_k} \sinh \frac{\gamma_k t}{2} \right)^{\frac{2\lambda_k \eta_k}{\theta_k^2}} \exp\left( -\frac{2y_0^0 u}{\lambda_k + \gamma_k \coth \frac{\gamma_k t}{2}} \right), \quad (7)
\]

where \( \gamma_k = \sqrt{\lambda_k^2 + 2\theta_k^2 u} \) and \( y_0^0 \) is the initial value of the integrated CIR process for sector \( k \). One merely needs to substitute \( u = -\xi_k(z) \).

Finally sector independence yields \( G_L(z) = \prod_{k=1}^K \mathbb{E}(\exp(\xi_k(z) \cdot Y_t)) \).

The Laplace transform, in particular \( \gamma_k \) above is not well defined for all values of \( z \). Hence we restrict the domain of \( z \) to \([0, 1]\) to guarantee that the Laplace transform exists. This does not pose any problem with the interpretation of the probability generating function \( G_L(z) \) as we are interested in derivatives of this function only as \( z \) approaches 0 from the positive side of the real line.

**PGF inversion** Thus the probability generating function for the portfolio loss is known explicitly. In principle, it should be straightforward to numerically compute derivatives approaching zero to yield loss probabilities. This is however not so straightforward as it seems
because numerical errors for higher derivatives in the neighborhood of zero are quite large and introduce significant error in the tail of the loss distribution. This is quite unfortunate, as it is the tail of the distribution that is of primary interest in the context of credit risk.

3.2.5 Characteristic Function approach

Advantages Given the numerical difficulties in inverting the portfolio loss PGF we modify the above approach slightly to work with the characteristic function instead of the PGF. This obviates the need to take higher order derivatives in the neighborhood of zero; we use Fourier transform techniques instead to obtain the loss distribution. This yields a numerically much more robust computation of the loss distribution. In particular the numerical accuracy for the tail of the distribution is not significantly worse than elsewhere in the distribution. Furthermore, an important additional advantage of this approach is that we can define the non-negative loss given default $\nu_A \in \mathbb{R}^+$ without resorting to exposure banding as required by CreditRisk+ i.e. losses can be treated as continuous random variables.
Formulation  The general relation between the probability generating function $G$ and the characteristic function $\Phi$ is the following:

$$\Phi_L(z) = G_L(e^{iz}).$$

(8)

Therefore, the characteristic function of the portfolio loss is approximately

$$\Phi_{L_t}(z) = \mathbf{E}\left(\exp\left(\sum_{k=1}^{K} \xi_k(z) \cdot Y_{k,t}\right)\right),$$

where $\xi_k(z) = \sum_A (e^{iz\nu_A} - 1) \omega_{Ak} p_A$. Utilizing the fact that (7) can be extended to the complex plane, we obtain $\Phi_{L_t}(z) =$:

$$\prod_{k=1}^{K} \frac{\exp\left(\frac{\lambda_k^2 \gamma_k t}{\theta_k^2}\right)}{\left(\cosh\frac{\gamma_k(z)t}{2} + \frac{\lambda_k}{\gamma_k(z)} \sinh\frac{\gamma_k(z)t}{2}\right)^2} \exp\left(\frac{2y_k^0 \xi_k(z)}{\lambda_k + \gamma_k(z) \coth\frac{\gamma_k(z)t}{2}}\right),$$

(9)

where $\gamma_k(z) = \sqrt{\lambda_k^2 - 2\theta_k^2 \xi(z)}$.

3.2.6 Model parameters

We conclude the description of the model with some notes regarding the interpretation and estimation of the model parameters. $p_A$ is the one-period (usually one-year) default probability. Hence, it can
be estimated as in the static (one-period) model. The factor weights can be taken from the static model as well if we assume that the proportional influence of different latent sector processes does not change over time.

Of more interest are the parameters of the CIR process and its integral. For the beginning we assume that the hazard rate of the default process should have a constant expectation. This implies that the CIR process as adjustment process has mean $\eta = 1$. This assumption can later be relaxed since there are cases where default hazard rates could be increasing or decreasing, but we omit these possibilities here. The normalization of the integrated CIR process by setting the one-period default probability to $p_A$ implies (because of $\eta = 1$) $y^0 = 1$. For these parameter values, the variance of the integrated process is given by

$$\var(Y_t) = \frac{\theta^2}{\lambda^2} \left( -\frac{3}{2} + \lambda t + 2e^{-\lambda t} - \frac{1}{2}e^{-2\lambda t} \right)$$

(10)

Parameter $\theta$ can be chosen such that the one-period variance of the integrated process matches the variance of the corresponding sector variable in the static model. Hence, we have one additional free parameter $\lambda$, namely the speed of mean reversion of the CIR process.
4 Illustrated Example

As illustrating example we use a portfolio consisting of 1000 obligors with average exposure of 1 and average default probability of 1%. The 5 industry factors are equally weighted, and the default values for the parameters of the factor processes are $y_0 = \nu = \sigma = \lambda = 1$. Substituting these values in the expression in equation 9 gives us the characteristic function of the portfolio loss. Inverting the characteristic function using Fast Fourier Transform is straightforward, following closely the steps outlined in Reiss 2004. This forms the base case analysis. The graphs in figures 1, 2 and 3 show how time, volatility and speed of mean reversion influence the loss distribution.

5 Conclusion

We proposed a parsimonious model for portfolio credit risk by extending the CreditRisk+ framework. In particular we introduced dynamic risk factors modeled using independent CIR processes. The figures using simulation results the illustrated example show clearly that both the expected loss and the variance of loss increase with time. An increase in mean reversion speed does not influence the expected
loss very much, while it decreases the variance of portfolio loss. The volatility $\sigma(t)$ of the CIR process decreases expected loss without affecting much the variance of portfolio loss. Therefore, we have with $\lambda$ and $\sigma$ two parameters which act somewhat independently on the loss distribution and can hence be used effectively to fit the model to data.

References


Figure 1: Evolution of the loss distribution over time
Figure 2: Influence of mean reversion speed on the loss distribution
Figure 3: Influence of factor process volatility on the loss distribution