TRANSACTION TAXES IN A PRICE MAKER/TAKER MARKET

ABSTRACT. We develop a price maker/taker model to study how a financial transaction tax affects markets where potential traders take a price or quote prices for the next potential trader. We find taxes widen quoted and effective spreads by many times the tax; significantly reduce asset prices, gains from quoting, gains from trade, and volume; and, do not eliminate destabilizing speculators. These effects are amplified in markets with intermediation. Volatility may decrease slightly without intermediation but increases significantly with intermediation. We find some early evidence of tax effects predicted by our model. JEL: D40, G18, G24, L10

Keywords: transaction tax, Tobin tax, market microstructure, limit order model, market makers, search costs

Taxing financial transactions is of perennial interest to regulators. A common view among tax supporters is that most short-term financial transactions are made by speculators causing excessive volatility in the financial markets; they posit that a tax would deter harmful speculation and raise significant tax revenues. Opponents of such a tax argue that it would increase price volatility and the cost of trading (bid-ask spreads) while reducing trading volumes and asset prices. They also claim that these effects would increase the cost of capital and that the overall effect might not be worth the putative benefits.

We model a price maker/taker market and then study how a transaction tax affects that market. This model gives economists a new tool to assess these claims and consider the effects of enacting, changing, or repealing a transaction tax. The model also yields insights into the effects of transaction taxes on markets with market makers (intermediaries). Since investors and intermediaries compete for liquidity, this work is especially relevant for markets with high-frequency traders or thin liquidity.¹

¹“Thin” liquidity means that there is little depth and the size of the inside bid-ask quotes are small.
Many supporters of a tax cite Tobin’s (1974) proposal of a 1% tax on foreign exchange transactions to reduce short-term speculation post-Bretton Woods.\footnote{This idea was reiterated and expounded upon in Tobin (1978) and Tobin (1984).} He hoped to “throw sand in the wheels of the market” by discouraging noise-creating speculation. However, this strategy was designed to achieve a policy objective: allowing nations greater leeway in exchange rate policies and in staving off monetary crises.

Our model is an innovative tool that can inform policy makers about the economic effects of transaction taxes. The model helps explain previously-observed effects of taxes, some of which have been puzzling to economists. For example, we show that taxes may reduce volatility even as they also reduce volume and increase spreads. Our approach allows us to evaluate the effects of a tax on quoted and effective spreads, volume, volatility, gains from quoting, gains from trade, deadweight loss, and asset prices as well as evaluate revenue-optimal taxes.

The model features a sequence of traders who strategically choose price taking versus price making, an approach that mirrors market behavior observed by Anand et al. (2005) and Hasbrouck and Saar (2009). Our model is similar to Foucault (1999) but allows for a range of private reserve valuations as well as varying proportions of pure market makers. These sources of variation enable us to study the impact of a transaction tax on market makers and investors. Since maker/taker market models pose many challenges, this is the first such model that addresses the effects of transaction taxes.

We first analyze the model theoretically which lets us prove certain properties and bounds for the model behavior. We then extend the analysis to a few specific cases. For these cases, we first consider a uniform distribution of reservation values which, while simple, allows us to find a closed-form solution and derive comparative statics. We then repeat the analysis with the addition of market makers. Finally, we consider a distribution of reservation values with market makers and normally-distributed investor types. We show there is a unique Markov perfect equilibrium and then analyze it numerically. We believe this analysis is
more relevant for current capital markets and thus more useful for quantifying potential effects of a transaction tax on markets.

The sequential market structure is most applicable to markets which are thin, *i.e.* they have very small quantities of securities at the bid and ask, or markets where there are many substitutes (such as corporate bonds). Black (1971) noted that markets in which traders compete on speed to get the best price are effectively thin; and, Chordia et al. (2011) shows that markets have become thin with the increase in high-frequency trading. Therefore, this model also has policy implications for markets with high-frequency traders. A nice feature of our model is that we can examine search costs with no assumption about the arrival rates of a match; rather, matching happens endogenously by traders setting prices to achieve their equilibrium maximum benefit.

We find that participants whose trades are taxed widen their quotes by more than four times the nominal tax, a considerably larger effect than we had anticipated. Those traders also derive less value from providing liquidity (quoting) and are less likely to trade. Our extended model that is closest to real capital markets suggests that for small values of the tax, a 1 basis point (bp; 0.01%) tax increase decreases the volume traded by 0.5%–1%; and, for taxes up to about 20 bp, each 1 bp increase in tax reduces asset prices by about 35 bp. The model also suggests that the maximal revenue raised is about half of the naïve assumption of tax \( \times \) pre-tax volume.

For a 50 bp tax, the model suggests that asset prices fall by 8%–11%; quoted spreads are about 3× untaxed spreads; effective (trade-realized) spreads increase by about 3× the nominal tax; volume, the expected benefit of providing liquidity, and gains from trade all fall by half; search costs almost double; and, the realized volatility is about 1.2× the volatility without a tax. If half of the arriving traders are market makers, the realized volatility increases monotonically versus the tax to more than triple for a 50 bp tax.

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3Sequential trader models are applicable to thin markets because at most one order may be filled by the depth present at the inside bid or ask. These models are applicable to markets with many substitutes because a potential trader does not come repeatedly to the market but only buys the cheapest of many substitutable assets.

4Search costs are a measure of liquidity defined by Lippman and McCall (1986) as “the time until an asset is exchanged for money.”
The only benefit we find is for volatility in markets where none of the participants are
market makers: in that case, the realized volatility decreases slightly up to a tax of 15 bp
— and then increases for taxes above 15 bp. The revenue-optimal tax of 57–69 bp increases
spreads by about half; reduces volume by about half; and, reduces the benefits of providing
liquidity by about two-thirds.

Taxes also do not reduce the effects (externalities) of destabilizing speculators; rather,
they accentuate those effects. As to the effects of intermediation in high-frequency or “thin”
markets, we find that replacing half of potential investors with market makers increases
the optimal spread by up to 25%. This effect occurs because investors and market makers
compete for liquidity. However, market makers reduce the volatility of trade prices. In
general, we find that market quality is increasingly sensitive to taxes with market makers.

1. Literature Review

Historically, securities transactions taxes have been proposed, enacted, modified, and even
repealed in various countries. An overview of the issues is given in Eichengreen et al. (1995).

Sau (1996), Palley (1999), and Baker (2000) have all proposed ‘Tobin-like’ taxes for various
financial markets. ul Haq et al. (1996) and Spahn (2002) suggest a 0.1% to 0.2% tax would
balance the opposing objectives of lowering price volatility due to speculation and maintain-
ing market liquidity. Pollin et al. (2003) consider a securities transaction tax for US financial
markets as “one feature of a new financial architecture aimed at contributing to financial
stabilization.”

Studies opposed to transaction taxes include Friedman (1953), Grundfest and Shoven
and Forbes (2001). Both ul Haq et al. (1996) and Spahn (2002) point out that their proposed
taxes would likely have little impact on speculative activity. (We agree based on our results
for destabilizing speculators as in De Long et al. (1990).) Kupiec (1996) indicates that a

5Countries which have considered or enacted financial transaction taxes include Australia, Brazil, China,
France, Germany, India, Japan, Singapore, Sweden, Taiwan, and the United Kingdom.
tax might lower price volatility but that it would also decrease asset prices such that return volatility increases with taxes. Schwert and Seguin (1993) give a comprehensive overview of arguments both for and against a securities transaction tax.

Empirical studies disagree as well. Umlauf’s (1993) study of Sweden imposing a 1% transaction tax in 1984 (and doubling it in 1986) found that 30% of equity trading volume moved to London, the market for interest rate options dried up, market volatility did not decline, and volume did not return to pre-tax levels when the tax was repealed in 1987.\(^6\) Liu and Zhu (2009) found the October 1999 deregulation of full commissions in Japan significantly increased price volatility in the equities market. Jones and Seguin (1997), however, found that reducing commissions on the New York Stock Exchange (NYSE) and American Stock Exchange (AMEX) in 1975 was followed by reduced market volatility in the following year.

Habermeier and Kirilenko (2001) posit three reasons for this disagreement. First, securities transaction taxes are often enacted with other policy shifts confounding causal inferences about changes in market measures. Second, measuring the tax reduction of noise trading is difficult since there is no way to determine if decreased market volume is due to informed traders or noise traders. Third, if asset prices change because of the tax, there is no way to determine if this was due to anticipation of the tax, trading moving to other venues, or untaxed securities. We would add two more reasons. First, comparing empirical studies is harder because securities transaction taxes vary in size and scope — from 0.13 bp to 528 bp. Second, reducing commissions is not the same as reducing a tax since lower commissions reduce rent-extraction and expose financial firms to more competition. Thus initial conditions of profitability and competition might explain the differences between Liu and Zhu (2009) and Jones and Seguin (1997). All of these issues make excellent arguments in favor of theoretical studies.

A few microstructure-based studies have been done of transaction taxes. Dupont and Lee (2007) investigated a transaction tax using a Glosten and Milgrom (1985) model incorporating spread and depth. They found that higher information asymmetry made the tax

\(^6\)For the years 1988–1990 between 48% and 52% of trading volume in Swedish equities occurred in London. This may explain Swedish opposition to a transaction tax.
more likely to decrease market liquidity. Mannaro et al. (2008) used heterogeneous agent
types to study transaction taxes in simulated markets. For a single market, they found
volatility increased as the number of orders decreased; for two competing markets, traders
tended to avoid the taxed market — which exhibited higher volatility than the untaxed
market. Cipriani and Guarino (2008) found that a tax caused a laboratory financial market
(sequential trading, one market maker) to cease trading during large disparities between an
asset’s price and true value. However, reduced noise trading caused by the tax offset some
of the induced market inefficiency. Pellizzari and Westerhoff (2009) showed that if adequate
liquidity were maintained a tax could help stabilize both double-auction and dealer mar-
kets.7 Otherwise, a tax reduced trading volume which increased volatility as each trade had
greater price impact. Demary (2010) used an agent-based framework to show that tax rates
above 0.1% destabilized the market and that taxes had stronger effects on more risk-averse
traders. Foucault and Colliard (2012) study the effect of markets setting access charges
and rebating fees to traders demanding or providing liquidity. While they explore numerous
conditions for equilibria, their model studies profit-maximizing limit order markets instead
of a government-imposed tax and does not consider markets with market makers. Amihud
and Mendelson (1986) studied the effect that bid-ask spreads have on asset prices, which
lets us show that taxes have a first-order effect on asset prices.

Our model also allows inference about search costs. Typically, search models assume
sequential search and bargaining and explicitly detail the search process (for example, as-
suming Poisson arrival rates to the market). Buyers and sellers seek to trade one unit of an
asset with pairings assigned by a matching process. Paired traders bargain in an attempt to
agree on a price of the asset and re-enter the market until an agreement is reached. Examples
of such models include Diamond (1982), Rubenstein and Wolinsky (1985), Gale (1987a,b),
Binmore and Herrero (1988), Lu and McAfee (1996), Mortensen and Wright (2002), Duffie
et al. (2005), and Duffie et al. (2010). Our model is more similar to how Lo and MacKinlay
(1990) looked at nonsynchronous trading.

7Maintenance of adequate liquidity for such findings to be relevant is more likely in a dealer market where
a market maker is expected to maintain a two-sided market.
2. Model of a Price Maker/Taker Market

Our limit order book model is similar to the simple price maker/taker model of Foucault (1999). The economy has one risky asset with fundamental value $v$. Traders arrive sequentially, one per unit of time, and have a spectrum of idiosyncratic reserve values $v + d_t$ where $d_t \sim F$, with support $\Omega$ where $\Omega = \mathbb{R}$ or some finite subset of $\mathbb{R}$. We assume that $F$ is a symmetric distribution with a mean of zero and finite variance: a trader with $d_t < 0$ would prefer to sell, a trader with $d_t > 0$ would prefer to buy. Each $d_t$ is that trader’s private information.

We assume that traders have heterogeneous reasons for trading: alpha (real or perceived), business risks to hedge, and inventory risk to eliminate being a few such reasons. We also assume traders have access to inventory or borrowed stock which allows them to sell without constraints. For simplicity and tractability, quotes live for only one period and the market continues at each period with non-unit probability $\rho$. Traders realize the utility of their trade (whether through expected return or by benefiting from a risk-reducing hedge) immediately following trading.

2.1. Strategic Quoting: Make or Take? Each trader seeks to trade one unit of the risky asset. Trading may be done by taking the prevailing bid-ask quote (i.e. sending a market order) or by making a new bid-ask quote (sending limit orders which replace the prevailing quote). Thus each trader plays a game against the next trader. To clarify the exposition, we will call these two traders Ilsa (the time $t$ trader) and Rick (the time $t + 1$ trader).

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8 Traders in the Foucault (1999) framework have only two possible reservation values, $v \pm L$ which occur with equal probability. We allow a range of private valuations and quotes based on these private valuations because for only two possible valuations, as in Foucault (1999), a tax would either have no effect or eliminate all trading.

9 Institutional traders generally have access to borrowable stock via brokerage customer’s holdings as well as market making inventory.

10 This yields phenomena seen in markets such as failure to trade when no one of the opposite preference arrives in the market. Since trading is not guaranteed, the model helps explain how equilibrium fill rates/volumes are affected by changes in market setup.

11 While we use this language to discuss the game played by each trader, this should not be construed as implying a repeated game. The setup is a sequential trader model: as time goes by, a sequence of unique traders arrive. Thus if a trader Sam first plays the game at any time $s$, we know that Sam will never play it again.
Traders pay tax at both position entry and exit: if Ilsa trades with Rick, each is debited a transaction tax of \( \tau \) on both entry and exit of their positions. Consequently, one could conjecture that if Ilsa quotes a bid and ask, she will shade her quotes, i.e. pass on some amount of the tax to Rick by quoting a bid of \( v - \delta_t \) and an ask of \( v + \beta_t \), where \( \delta_t \) and \( \beta_t \) are functions of the tax \( \tau \). Ilsa solves for these equilibrium offsets to decide the optimal amount of the tax to (potentially) pass on to Rick. This strategic price shading causes \( \delta_t \) and \( \beta_t \) to be functions of Ilsa’s reservation value \( v + d_t \), the distribution of Rick’s unknown reservation value \( v + d_{t+1} \), and the transaction tax \( \tau \). A diagram of Ilsa’s possible gains is shown in Figure 1.

![Diagram](image)

**Figure 1.** Ilsa’s expected gains in different scenarios. Her known gain for price taking is \( R_T|d_t \); her expected gain for quoting optimal prices is \( R^*_Q|d_t \). With probability \( \rho \leq 1 \), the market continues.

Because the game played every time period is between the current and next-period trader, the solution takes a form which does not depend on past states. Therefore, we can assume that prices are static (i.e. \( \Delta v = 0 \)) without loss of generality and solve for equilibrium \( \delta \) and \( \beta \) (without time subscripts). Were \( \Delta v \neq 0 \), we could merely shift the \( d_t \) distribution (as well as \( \delta \) and \( \beta \)) by \( E(\Delta v) \). Were \( \text{Var}(\Delta v) > 0 \), we could scale the \( d_t \) distribution to have a variance of \( \text{Var}(\Delta v) + \text{Var}(d_t) \). Therefore, while we work with static prices, the results here are applicable to stochastically-evolving prices.

We determine optimal trade strategies by working forward from time \( t \). The time \( t \) trader, Ilsa, has imperfect knowledge: She sees her reservation value \( v + d_t \) but not Rick’s \( (v + d_{t+1}) \); however, the distribution of reservation values is common knowledge. Taking the quoted bid
or ask price has a known benefit of price taking $R_T|d_t$,

$$R_T|d_t = \max(d_t - \beta_{t-1} - 2\tau_t, -d_t - \delta_{t-1} - 2\tau_t).$$

(1)

If Ilsa decides to quote, she chooses not to take the current bid or ask price for benefit $R_T|d_t$; she prefers to quote optimal bid and ask prices for the time $t + 1$ trader, Rick, for an expected conditional benefit of $R^*_Q|d_t$:

$$R^*_Q|d_t = R_Q(\delta^*, \beta^*)|d_t = \max_{\delta, \beta} R_Q(\delta, \beta)|d_t,$$

(2)

where $\delta^*$ and $\beta^*$ are the optimal bid and ask offsets that maximize Ilsa’s expected quote revenue $R_Q|d_t$, given later in Equation (7).

This optimal expected value defines the boundary between sending a market order and quoting and varies with the reservation value $v + d_t$. Her utility is then the greater of the known and expected benefit:

$$U_t = \max(R_T|d_t, R^*_Q|d_t).$$

(3)

When the next trader, Rick, enters the market at time $t + 1$, he decides whether to trade against Ilsa’s quote or quote a bid and ask for the following trader. While Ilsa does not know Rick’s reservation value $v + d_{t+1}$, she can find the unconditional expectation of his optimal quote revenue which we will formally define and solve for later.

Ilsa chooses the better of $R_T|d_t$ and her (optimal) $R^*_Q|d_t$. (Her actions only affect $R_Q|d_t$.) Ilsa’s expectation of Rick’s expected benefit of quoting is:

$$R^*_Q = E[(R_Q(\delta^*, \beta^*)|d_{t+1})] = \int_{\Omega} \max_{\delta, \beta} R^*_Q(\delta, \beta)|d_{t+1}dF.$$

(4)

Since all reservation values ($d_t$’s) are iid, $R^*_Q$ is the equilibrium expected benefit of quoting.

2.2. Market Quality. To assess the effects of a tax on market quality, we examine the following metrics:

**Optimal Spread:** Revenue-maximizing bid-ask spread.
Quoted Spread: Optimal spread, when quoted.\textsuperscript{12}

Effective Spread: Average difference between buy, sell prices.

Fill Rate: Fraction of orders which trade; analogous to volume.

Search Cost: Expected time until an order is filled.

Expected Quote Revenue: $R_Q^0$; equilibrium expected quote revenue.

Realized Volatility: Volatility of trade price changes.

Gains from Trade: $R_T|_{\text{trade}}$; realized gains of price taking.

Deadweight Loss: Gains from trade + tax revenue vs without tax.

Revenue-optimal Tax Rate: Tax rate which maximizes tax revenue.

Stabilization: Reduction in trading by destabilizing traders.

Asset Prices: Percent increase/reduction in asset prices.

For many of these metrics, we look at the equilibrium expected value. While most of them are easily found by solving the game played by Ilsa and Rick, in some cases we must find the quoted spread, effective spread, realized volatility, gains from trade, and deadweight loss via simulation.\textsuperscript{13} In those cases, we iterate over tax levels; for each level of the tax, we generate a random sequence of traders with iid reservation value offsets $d_t$’s and record their trading decision (make or take prices). If a trade occurs, we record the direction (buy/sell), the price, and the expected gain from trade.

2.3. Characterizing Propositions. With the game well-defined, there are a number of properties of the game we can investigate. We characterize the game by proving a number of propositions. (Proofs are in Appendix A.)

We return to Ilsa’s decision: She must take the existing quote or make a new quote for Rick. Rick is in the same position as Ilsa: he does not know the type of the following trader (Sam) and so makes his decision according to the logic in Table 1. We can also show that one and only one of the clauses in Table 1 will always be satisfied

\textsuperscript{12}The difference between the optimal and quoted spread arises from strategic choice. A trader always computes an optimal quote, but may choose to take the preceding prices instead of quoting their optimal quote.

\textsuperscript{13}While we could try to derive the volatility from the quoted spread using Roll (1984), that would ignore the endogeneity of when trade occurs and thus would be inaccurate.
<table>
<thead>
<tr>
<th>Condition</th>
<th>Action</th>
</tr>
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<tbody>
<tr>
<td>(v - \delta - (v + d_{t+1}) - 2\tau &gt; R^*_Q</td>
<td>d_{t+1})</td>
</tr>
<tr>
<td>(\implies d_{t+1} &lt; -R^*_Q</td>
<td>d_{t+1} - \delta - 2\tau)</td>
</tr>
<tr>
<td>(v + d_{t+1} - (v + \beta) - 2\tau &gt; R^*_Q</td>
<td>d_{t+1})</td>
</tr>
<tr>
<td>(\implies d_{t+1} &gt; R^*_Q</td>
<td>d_{t+1} + \beta + 2\tau)</td>
</tr>
<tr>
<td>Otherwise</td>
<td>Quote new bid and ask for Sam</td>
</tr>
</tbody>
</table>

Table 1. Rick’s decision rules for taking Ilsa’s quoted prices. If none of the conditions are satisfied, Rick will quote his own bid and ask instead.

**Proposition 1.** If taxes, bid-ask spreads, and the optimal quote revenue \(R^*_Q\) are non-negative, then Rick (and all traders) have a clear course of action: only one of the conditions in Table 1 is satisfied.

Under the preceding conditions, Ilsa’s conditionally expected quote revenue, \(R_Q(\delta, \beta)|d_t\), for an arbitrary quoting strategy is:\(^{14}\)

\[
R_Q(\delta, \beta)|d_t = \rho P(\text{Rick sells at bid})(v + d_t - (v - \delta) - 2\tau) + \rho P(\text{Rick buys at ask})(v + \beta - (v + d_t) - 2\tau)
\]

\[
= E[\rho F(-R^*_Q|d_{t+1} - \delta - 2\tau)(\delta + d_t - 2\tau)|d_t]
\]

\[
+ E[\rho (1 - F(R^*_Q|d_{t+1} + \beta + 2\tau)) (\beta - d_t - 2\tau)|d_t]
\]

\[
= \rho F(-R^*_Q - \delta - 2\tau)(\delta + d_t - 2\tau) + \rho (1 - F(R^*_Q + \beta + 2\tau)) (\beta - d_t - 2\tau).
\]

Ilsa attains her maximum expected quote revenue, \(R^*_Q\) by setting \(\frac{\partial R_Q}{\partial \delta}\) and \(\frac{\partial R_Q}{\partial \beta}\) to 0 and solving for \(\delta^*\) and \(\beta^*\). Ilsa’s optimal strategy is to bid at \(v - \delta^*\) and ask for \(v + \beta^*\). The

\(^{14}\)We invoke Fubini’s Theorem to exchange the integration operators.
bid and ask offsets satisfy:\(^{15}\)

\[
\delta^* = \frac{F(G(\delta^*))}{f(G(\delta^*))} - d_t + 2\tau, \quad \text{and} \quad \beta^* = \frac{F(G(\beta^*))}{f(G(\beta^*))} + d_t + 2\tau, \quad \text{where}
\]

\[
G(x) = -R_Q^d - x - 2\tau \quad (\text{expected gain boundary})
\]

defines the boundary separating where Ilsa expects Rick would gain by trading with her.\(^ {16}\)

Specifically, if Rick’s \(d_{t+1} < G(\delta)\), Ilsa expects that Rick would gain by selling to her at her bid; and, if Rick’s \(d_{t+1} > -G(\beta)\), Ilsa expects Rick would gain by buying from her at her asking price.

Consistent with our intuition, we can see that the bid and ask are skewed in line with Ilsa’s reservation value: For a positive \(d_t\), she will quote a bid and ask higher than for \(d_t = 0\); and, for a negative \(d_t\) she will quote a bid and ask lower than for \(d_t = 0\).

**Proposition 2** (Bid and Ask Skewed with \(d_t\)). If \(d_t > 0\), then the optimal bid and ask parameters are such that \(\beta^* \geq \delta^*\). Similarly, if \(d_t < 0\), then the optimal bid and ask parameters are such that \(\beta^* \leq \delta^*\).

We can also put a lower bound on the bid-ask spread. The lower bound shows that in all situations, the spread is more than four times the nominal tax rate.

**Proposition 3** (Spread Exceeds 4× Nominal Tax). For any distribution \(F\) of reservation values, the optimal bid-ask spread \(\delta^* + \beta^*\) (and thus the quoted and effective spread) is greater than \(4\tau\): i.e. \(\delta^* + \beta^* \geq 4\tau\). Put differently: all spread measures are more than four times the quoted tax rate, twice the taxes collected from any trader or at any time, and more than the total round-trip taxes for all involved traders.

With the lower bound, we can also show that the benefit of quoting a bid and ask for all players is always positive.

\(^{15}\)Note that the optimal bid and ask offsets, \(\delta^*\) and \(\beta^*\) are functions of \(d_t\) which is suppressed for notational simplicity. Furthermore, we consider only those \(\delta^*\) and \(\beta^*\) such that \(f(\delta^*) > 0\) and \(f(\beta^*) > 0\).

\(^{16}\)Note that we have used the symmetry of \(F\) to express the probability of Rick trading at Ilsa’s asking price in equation (7) via \(F\) instead of \(1 - F\).
**Proposition 4** (Positive Expected Quote Revenue). *For any player (of type $d_t$), the expected quote revenue is always positive at the optimal bid and ask: i.e. $R^*_Q|d_t > 0$.*

Finally, with some very general assumptions, we may show that there exists a unique Markov Perfect equilibrium for the game between the current and next-period trader.

**Proposition 5** (Existence of Markov Perfect Equilibrium). *Assume the preceding setup with bounded quotes and taxes. If the pdf of $d_t$’s, $f$, is such that $d_t$ has finite variance, the game played between Ilسا and Rick (or any two subsequent traders) has a Markov Perfect equilibrium.*

**Proposition 6** (Unique Markov Perfect Equilibrium: Continuous CDF). *Assume the preceding setup with bounded quotes and taxes and that the cumulative distribution function of $d_t$’s is continuous. Then there is a unique Markov Perfect equilibrium of the game played between Ilسا and Rick (or any two subsequent traders).*

The last two propositions mean that we can expect our model to be solvable if the distribution of market participants’ private reservation values is neither discrete nor extremely heavy-tailed.

### 3. Uniform Investors

We first examine the model behavior for two simple distributions of reservation values. This gives us insight into the model and shows that solving the model does not require a particular distributional assumption. The simplicity of the distributions also yields a closed-form for the solution and allows us to derive comparative statics for metrics of market quality.

The first simple distribution we consider is a market with only investors; their reservation value offsets are uniformly-distributed: $d_t \sim \text{Unif}[-L, L]$. Since we are also interested in the effect of a tax on markets with intermediation, the second simple distribution is a mixture of the first (uniform) distribution with a point mass at $d_t = 0$ to represent the indifference of market makers between buying and selling.
3.1. Uniform Investors Without Intermediation. We first consider a market with only investors, i.e. being indifferent between buying and selling is a measure-zero event. The reservation value offsets are uniformly-distributed: $d_t \sim \text{Unif}[-L, L]$. This lets us find the equilibrium expected quote revenue as well as the optimal bid, ask, and spread. From there, we can find other metrics and comparative statics. All proofs are in Appendix B.

**Proposition 7 (Metrics of Market Quality I).** The optimal bid, ask, and spread for uniform investors without intermediation is $\delta^* = \frac{1}{2}(L - R_Q^\circ - d_t)$, $\beta^* = \frac{1}{2}(L - R_Q^\circ + d_t)$, and $L - R_Q^\circ$, where the expected quote revenue $R_Q^\circ$ satisfies the equation

\begin{equation}
R_Q^\circ = \frac{\rho}{24L^2}(2L - R_Q^\circ - 4\tau)^3
\end{equation}

Since the optimal spread is constant, the quoted spread is also $L - R_Q^\circ$.

**Proof.** Use equations (8)–(9). Details are in Appendix B. \qed

Since the RHS of equation (10) has a triple root, $R_Q^\circ$ has a unique solution.

**Corollary 1 (Comparative Statics).** In equilibrium:

(1) increasing taxes reduces the value of quoting (providing liquidity);
(2) increasing the likelihood the market stays open increases the value of quoting;
(3) increasing taxes increases bid-ask spreads; and,
(4) increasing the likelihood the market stays open decreases spreads.

**Proof.** Here, we sketch the proof. Differentiating $R_Q^\circ$ with respect to tax $\tau$ or continuation probability $\rho$, we get

\begin{equation}
\frac{\partial R_Q^\circ}{\partial \tau} = -\frac{\rho(2L - R_Q^\circ - 4\tau)^2}{2L^2 + \frac{\rho}{4}(2L - R_Q^\circ - 4\tau)^2} < 0,
\end{equation}

\begin{equation}
\frac{\partial R_Q^\circ}{\partial \rho} = \frac{(2L - R_Q^\circ - 4\tau)^3}{24L^2 + 3\rho(2L - R_Q^\circ - 4\tau)^2} > 0.
\end{equation}

Since the optimal spread is $L - R_Q^\circ$, we know the effects on the spread are of opposite sign. \qed

Furthermore, we can derive other related metrics of market quality.
Proposition 8 (Metrics of Market Quality II). The effective spread, fill rate, search cost, and the effect on asset prices are given by:

1. The effective spread is the difference $E(Ilsa’s \text{ sell price}|Ilsa \text{ sells}) - E(Ilsa’s \text{ buy price}|Ilsa \text{ buys})$
   
   $$L - R_Q^\circ + \frac{1}{2L} \left[ \frac{(R_Q^\circ + 4\tau)^2}{8} - \frac{L^2}{2} \right],$$
   which increases with $\tau$.

2. The expected fill rate of an order is given by $E[\text{Prob(buy)} + \text{Prob(sell)}] = \frac{[2L-(R_Q^\circ + 4\tau)]^2}{8L^2}$
   which decreases with $\tau$.

3. We define the expected search cost $C_S$ to be the time it takes to fill an order. $C_S = \frac{8L^2}{[2L-(R_Q^\circ + 4\tau)]^2} \left( \frac{8L^2}{[2L-(R_Q^\circ + 4\tau)]^2} - 1 \right)$ increases with $\tau$.

4. The expected asset price $E[p] = \frac{[2L-(R_Q^\circ + 4\tau)]^2v}{8L^2}$ decreases with $\tau$.

Corollary 2. The effective spread is less than the optimal and quoted spread.

The uniform distribution allows us to derive the above metrics of market quality in closed form and study their behavior with variation in taxes. However, the uniform distribution has bounded private valuation offsets and so might not be as realistic as we hope for a model.\footnote{For example, under the uniform distribution the optimal spread and the quoted spread are the same and constant.} Therefore, we study normally-distributed private valuation offsets in Section 4. The tradeoff, however, is that we lose the mathematical tractability and some results must be found numerically. Reassuringly, the results derived here are consistent with those from numerical simulations.

3.2. Uniform Investors With Intermediation. We next consider investors with uniformly-distributed reservation value offsets: $d_t \overset{iid}{\sim} \text{Unif}[-L, L]$. We add market makers to the market by mixing the prior distribution of total mass $1 - \mu$ (representing investors) with a point mass of $0 < \mu < 1$ at $d_t = 0$ (representing the preferences of market makers).\footnote{Assigning $d_t = 0$ makes market makers indifferent between buying and selling; the effect of inventory risk is ignored. However, market makers with inventory risk can be thought of as belonging to the population of traders with non-zero $d_t$’s.} That is:

\begin{equation}
(13)
\begin{align*}
d_t \overset{iid}{\sim} \begin{cases} 
  0 & \text{w.p. } \mu \\
  \text{Unif}[-L, L] & \text{w.p. } 1 - \mu.
\end{cases}
\end{align*}
\end{equation}
This means that the $d_t$ CDF is no longer continuous. Therefore, we should expect the results to differ (and the work to be more involved). Supporting lemmas and proofs are in Appendix C.

We define the expected quote revenue $R_Q|d_t$ as

\begin{align}
R_Q|d_t &= \rho V_1(\delta) + \rho V_2(\beta) \quad \text{where} \\
V_1(\delta) &= [(1 - \mu)F(G(\delta)) + \mu I(G(\delta))](\delta + d_t - 2\tau) \quad \text{and} \\
V_2(\beta) &= [(1 - \mu)F(G(\beta)) + \mu I(G(\beta))](\beta - d_t - 2\tau).
\end{align}

$V_1(\delta)$ is the expected benefit of quoting a bid $v - \delta$ and $V_2(\beta)$ is the expected benefit of quoting an offer $v + \beta$.

Using these definitions, we can show that quoting may provide positive benefit, that is, \( \max_\delta V_1(\delta) > 0 \) and \( \max_\beta V_2(\beta) > 0 \) iff \( d_t \in (R_Q^c + 4\tau - L, L] \) and \( d_t \in [-L, L - R_Q^c - 4\tau) \), respectively.

**Lemma 1** (Positive Quoting Benefit). \( \max_\delta V_1(\delta) > 0 \) iff \( d_t \in (R_Q^c + 4\tau - L, L] \); \( \max_\beta V_2(\beta) > 0 \) iff \( d_t \in [-L, L - R_Q^c - 4\tau) \).

In the discussions that follow, we focus our attention on the region where the maximal expected quote benefit is strictly positive: a trader will make or take prices. Outside of that region, the trader will either take the market order or will not trade. In other words, we make two assumptions: That the round-trip tax does not exceed the expected benefit of trading for all investors; and, that the equilibrium quote revenue is less than the maximum round-trip quote revenue of trading with a most extreme buyer and seller.

**Assumption 1.** *Round-trip Tax* < *Max E(Gains to Trade)*: \( 2\tau < L \);

**Assumption 2.** *Equilibrium Quote Revenue* < *Max*: \( R_Q^c < 2L - 4\tau \).

The possibility that the market maker may not trade creates a discontinuity for the expected quote benefit, i.e. $V_1(\delta)$ and $V_2(\beta)$ are discontinuous at $\delta_0 = -R_Q^c - 2\tau$ and $\beta_0 = -R_Q^c - 2\tau$. 
Furthermore, evaluating the expected gain boundary $G$ at these boundaries yields $G(δ_0) = G(β_0) = 0$ and thus defines the boundaries of where a market-making Rick will trade at Ilsa’s bid or ask.

In Appendix C, we derive the partial derivatives $\frac{∂V_1(δ)}{∂δ}$ and $\frac{∂V_2(β)}{∂β}$. We may then easily verify that the second derivatives are non-positive:

**Lemma 2.** For all $δ, β ∈ \mathbb{R}$, $d_t ∈ [-L, L]$, $μ ∈ (0, 1]$, $R_Q^0 > 0$, and $τ ≥ 0$:

$$\frac{∂^2 V_1(δ)}{∂δ^2} ≤ 0, \text{ and } \frac{∂^2 V_2(β)}{∂β^2} ≤ 0.$$

With the first and second order derivatives, we can analyze the various cases for the functional behavior of $V_1(δ)$ in Figure 2 below. The behavior for $V_2(β)$ is similar (albeit for $-d_t$).

![Figure 2](image-url)

**Figure 2.** Three possible cases for $V_1(δ)$, the expected benefit of quoting a bid of $v - δ$ as a function of $δ$. Case I: $V_1(δ)$ jumps up at $δ_0$; Case II: $V_1(δ)$ is continuous; or, Case III: $V_1(δ)$ jumps down at $δ_0$. The cases are partitioned depending on the reservation price $v + d_t$ of the liquidity provider. The dot at $δ_0$ denotes that $V_1(δ)$ is left-continuous.

In Figure 2, $G(δ) > 0$ denotes the region in which a market maker will trade at Ilsa’s quote. We see that a market maker will trade only in Case III, when $4τ + R_Q^0 < d_t ≤ L$. In this case, we compare $V_1(δ_0)$ with the local maximum $V_1(δ_M)$ to find the global maximizer. Intuitively, when Ilsa’s reservation value is high, or the tax is low, her quote might appeal to a market maker or an investor. In contrast, a market maker will not take Ilsa’s quote in case I and II. In other words, Ilsa needs to wait for a non-market-maker to trade with her. As we increase taxes, investors will be pushed out of case III into cases I and II. If there are
a lot of market makers, the possibility that a trade will happen is lowered, or the spread has
to be further widened. In other words, in a market with intermediaries, a transaction tax
has a bigger effect.

We may find unique local maximizers $\delta_M$ and $\beta_M$ of $V_1(\delta)$ and $V_2(\beta)$, where $\frac{\partial V_1(\delta)}{\partial \delta}|_{\delta_M} = 0$, and $\frac{\partial V_2(\beta)}{\partial \beta}|_{\beta_M} = 0$. We explicitly derive the closed form solution for $\delta_M$, $\beta_M$ and the expected
benefits $V_1(\delta_M)$ and $V_2(\beta_M)$ in Corollary 6 in Appendix C.

It follows that $\max_\delta V_1(\delta) = \max\{V_1(\delta_0), V_1(\delta_M)\}$ and $\max_\beta V_2(\beta) = \max\{V_2(\beta_0), V_2(\beta_M)\}$. Consistent with the intuition
above, as the fraction of market makers approaches 1, Ilsa is better off providing a quote at
which a market maker will trade, $v - \delta_0$, effectively reducing the spread.

If the tax is excessively high, the value of quoting at the gain boundary ($G(\delta_0) = 0$) for a
market maker will be negative. In other words, a quote Ilsa expects a market-making Rick
to be indifferent to taking does not have expected benefit to her. Specifically, if $4\tau \geq L$, or
$4\tau > L$ but $R^*_Q + 4\tau \geq L$ then $V_1(\delta_0) \leq 0$ and $V_2(\beta_0) \leq 0$. If we instead assume $4\tau < L$
and $R^*_Q < L - 4\tau$, we get that Ilsa expects positive benefit to quoting a bid and ask at
the gain boundary for Rick. This is also the bid and ask most attractive to market makers;
however, “most attractive” does not mean they will find taking those prices more attractive
than quoting.

For Ilsa to expect a market-making Rick to trade against Ilsa’s quote, we require more
restrictive conditions: the expected value of quoting and the tax must be sufficiently small;
and, Ilsa’s private reservation price must be sufficiently extreme. In particular, Lemma 5 in
Appendix C shows that a market-making Rick will trade against Ilsa’s quote if

1. $0 \leq \tau < \frac{\mu L}{\sqrt{2\mu + 2\mu^2 + 2\mu}}$, (tax sufficiently small)
2. $R^*_Q \leq \frac{4\mu L}{\sqrt{2\mu + 2\mu^2 + 2\mu}} - 4\tau$, and (quote benefit sufficiently small)
3. $|d_1| > x_1 + R^*_Q + 4\tau$ (Ilsa’s view sufficiently extreme)

where $x_1 = \frac{(1-\mu)L}{1+3\mu+2\sqrt{2\mu+2\mu^2}}$.

\(^{19}\)For $d_1 > R^*_Q + 4\tau$. 

We can show there is a unique fixed-point expected quote revenue, $R^o_Q$ for the case with market makers — and that comparative statics for this market stay the same as for the uniform case without market makers.

**Proposition 9.** There is a unique equilibrium expected quote revenue $R^o_Q \in (0, 2L - 4\tau)$ such that $E[R_Q|d_t] = R^o_Q$. Furthermore, $\frac{\partial R^o_Q}{\partial \tau} < 0$, and $\frac{\partial R^o_Q}{\partial \rho} > 0$.

In other words: taxes reduce the expected equilibrium benefit of providing liquidity and a market more likely to continue trading increases the expected equilibrium benefit of providing liquidity.

The comparative statics for the proportion of market makers are a bit more involved. We first restate the conditions for finding the sign of $\frac{\partial R^o_Q}{\partial \mu}$.

**Proposition 10.** Suppose $R^o_Q$ is the solution to $E[R_Q|d_t] = R^o_Q$. Then the sign of $\frac{\partial R^o_Q}{\partial \mu}$ is the same as

\[
Z(\mu) = 3\left(1 - \frac{R^o_Q + 4\tau}{L}\right)^2 + 1 - \alpha^2(\mu)\left(\frac{\sqrt{2\mu + 2\mu^2}}{\mu(\sqrt{2\mu + 2\mu^2} + 2\mu)} + 1\right)
\]

where $\alpha(\mu) = \frac{4\mu}{\sqrt{2\mu + 2\mu^2 + 2\mu}}$. Furthermore, this sign depends on $\mu$ and $\rho$.

This suggests that the comparative statics need some qualification.

**Corollary 3.** For sufficiently high values of providing liquidity, increasing the proportion of market makers decreases the value of providing liquidity. Mathematically: if $(1 - \frac{1}{\sqrt{6}})L < R^o_Q + 4\tau < \alpha(\mu)L$, then $\frac{\partial R^o_Q}{\partial \mu} < 0$.

**Corollary 4.** For sufficiently low values of providing liquidity, increasing the proportion of market makers increases the value of providing liquidity. Mathematically: if $0 < R^o_Q + 4\tau < (1 - \frac{1}{\sqrt{6}})L$, then $\exists \mu > \mu_0$ such that when $\mu \geq \bar{\mu}$, $\frac{\partial R^o_Q}{\partial \mu} > 0$.

These results suggest a future extension of this model: exploring the equilibrium proportion of market makers.
4. Normal Investors

To study how a transaction tax affects market quality for a setup closer to real capital markets, we analyze traders in a market with private reservation values distributed according to a mixture of: (i) a point mass of $0 < \mu < 1$ at $d_t = 0$ (representing market makers), and (ii) a normal distribution with mean zero, variance $L^2$, and total mass $1 - \mu$ (representing investors):

\begin{equation}
    d_t \sim f = \begin{cases} 
    0 & \text{w.p. } \mu \\
    N(0, L^2) & \text{w.p. } 1 - \mu.
    \end{cases}
\end{equation}

We examine the model for tax $\tau$ ranging from 0 to 50 bp. We also study the effect of taxes for different proportions $\mu$ of arriving traders being market makers. The mean reservation value, $v = $20, is calibrated to be close to the average stock price in the US equity market. The reservation value standard deviation of $L = $0.50 (2.5% of $v$) is calibrated to be the same as the daily volatility of a stock with a 40% annualized volatility; and, the market continuation probability is set to be $\rho = 0.9$.\footnote{Results are robust to changes in $\rho$ since $\rho$ falls out of the optimal bid and ask formula.} We look at both the average effect across the market and the effect for market makers.

4.1. Existence and Uniqueness of Equilibrium. Ilsa’s conditional expected quote revenue for an arbitrary quoting strategy, $R_Q|d_t$, is:

\begin{equation}
    R_Q|d_t = \rho(1 - \mu)\Phi\left(\frac{-R_Q^0 - \delta - 2\tau}{L}\right)(\delta + d_t - 2\tau) \\
    + \rho(1 - \mu)\Phi\left(\frac{-R_Q^0 - \beta - 2\tau}{L}\right)(\beta - d_t - 2\tau) \\
    + \rho\muI(-R_Q^0 - \delta - 2\tau \geq 0)(\delta + d_t - 2\tau) \\
    + \rho\muI(-R_Q^0 - \beta - 2\tau \geq 0)(\beta - d_t - 2\tau)
\end{equation}

where $\Phi$ is the standard normal cdf. We then show that this distribution also admits a unique Markov Perfect equilibrium. (All supporting proofs are in Appendix D.)
Proposition 11 (Unique Markov Perfect Equilibrium: Normal with Market Makers). Under the preceding setup and for bounded quotes and taxes, the game played between Ilsa and Rick (or any two subsequent traders) has a unique Markov Perfect equilibrium.

Finally, these more specific assumptions about \( d_t \) allow us to find a stronger upper bound for the bid-ask spread:

Proposition 12 (Spread Upper Bound: Normal with Market Makers). If the distribution \( F \) of reservation values is \( N(0, L^2) \), the bid-ask spread \( \delta + \beta \) is bounded above by: 
\[
\delta + \beta \leq \frac{L}{R_Q^o + 4\tau} + 4\tau.
\]

4.2. Solving for Equilibrium. Since the iid distribution of all trader’s reservation values is common knowledge, Ilsa solves for expectation of Rick’s optimal fixed-point quote revenue \( R_Q^o \) given the unconditional distribution of \( d_{t+1} \) and maximizes \( R_Q^o | d_t \) in equation (20).\(^{21}\)

The one complication of our distribution of potential traders is the point mass of market makers at \( d_t = 0 \). Because of this, \( R_Q^o | d_t \) contains two indicator functions to handle the possibility that Rick is a market maker: one for the possibility that Ilsa’s bid appeals to a market maker and one for the possibility that her ask appeals to a market maker.\(^{22}\) We then find the constrained maxima over three regions: where neither indicator function is active, where the first indicator function is active, and where the second indicator function is active. (We know both indicator functions are never simultaneously active by Proposition 1.)

4.3. Metrics of Market Quality. Figure 3 shows that the dynamics for a market with static \( v \) and no taxes appear similar to that seen in real data. We can also note that having more market makers tends to stabilize the range in which a security trades.

While these plots reassure us that nothing is drastically wrong, visual examination is not a proper assessment of changes in market quality. We therefore examine the previously-defined metrics of market quality to assess the effects of the tax. For the metrics which

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\(^{21}\)In practical terms, solving for \( R_Q^o \) requires iterating over a sufficiently large range of values for \( d_{t+1} \); computing each optimal \( R_Q^o | d_{t+1} \); and then determining the unconditional expectation of \( R_Q^o | d_{t+1} \). We must make sure not to double-count the center of the distribution and handle the tails appropriately. We repeat this until \( R_Q^o \) has converged.

\(^{22}\)Market makers have the same reservation value and so would not trade with one another.
Figure 3. Simulated trades (stars) and quotes (lines) for a market with no transaction tax. The left plot is for a market with no market makers; the right plot is for a market with 50% of potential traders being market makers. The market with market makers trades in a tighter range: All trades occur between $19.75 and $20.25; trades occur outside this range for the market without market makers (left).

To require simulation, we generate a random sequence of 100,000 traders with $d_t \sim N(v, L)$ reservation value offsets.

4.4. Optimal and Quoted Spread. The left plot in Figure 4 shows that the optimal spread one would consider quoting increases with the tax: by 270% without market makers, from 65 bp with no tax to 240 bp with a 50 bp ($0.10$) tax; and, by 170% with half of potential traders being market makers, from 90 bp with no tax to 240 bp with a 50 bp tax. At $\tau = 50$ bp, the change in the optimal spread is $3.5 \times$ and $3 \times$ the change in the tax, for markets without market makers and with 50% market makers. In other words: traders pay 200%–250% more than the tax.

The right plot in Figure 4 shows the quoted spread — i.e. the optimal spread filtered by when traders choose to quote it instead of taking the prior quote. Regardless of the prevalence of market makers, adding a 50 bp tax approximately triples the quoted spread. The quoted spread without a tax is slightly higher for increasing proportions of market makers. This is because an increasing prevalence of market makers makes quoting less appealing for investors and often gives them better prices to take.
4.5. **Effective Spread.** Figure 5 shows that regardless of the prevalence of market makers, adding a 50 bp tax approximately triples the effective spread. (The effective spread without a tax is slightly lower for increasing proportions of market makers.) This differs from the results for the optimal spread in that the optimal spread increased with higher proportions of market makers. This difference is because an increasing prevalence of market makers makes quoting less appealing for investors and often gives them better prices to take.

4.6. **Fill Rate and Search Cost.** Figure 6 shows two plots: the left plot shows the fill rate and the right plot shows the fill rate adjusted for the fact that market makers do not trade with one another. In both plots we see a 50% drop in the fill rate (volume) at a tax of 50 bp regardless of the prevalence of market makers. The lower fill rates for increasing proportions of market makers is largely due to the fact that market makers do not trade with one another in this model. For example, were we to ignore the times when one market maker follows another, the untaxed fill rate for half the arriving traders being market makers would be 72% — much closer to the 75% fill rate without market makers.  

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23The 72% figure is found by dividing the untaxed fill rate of 54% by the probability of not having two market makers in a row (0.75).
Figure 5. Effective spreads (spreads at trade times) vs transaction tax rate (bp). Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). More market makers reduce effective spreads slightly and higher taxes increase effective spreads.

Figure 6. Fill rate (left) and adjusted fill rate (right) vs transaction tax rates (bp). The adjusted fill rate corrects for the fact that market makers do not trade with one another. Curves are shown for 10% increments of market makers, from none to half of arriving traders being market makers. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). The fill rate drops by about half for a tax of 50 bp.

Harris (2002) says, “Trading is a search problem. […] Sellers seek buyers willing to pay high prices. Buyers seek sellers willing to sell at low prices.” While search models often take the form of sequential search and bargain models, our model can yield insight into search
costs if we focus only on how long it takes in toto until a trade occurs. Since Lippman and McCall (1986) view liquidity as “the time until an asset is exchanged for money”, we study how a transaction tax affects the average time between trades. If the probability of a fill is $P_f$ (i.e. the fill rate), we can infer relative search costs (waiting times), $t_w$, by inverting the fill rate: $t_w \propto P_f^{-1}$. Search costs are shown in Figure 7.

\begin{figure}[h!]
\centering
\includegraphics[width=0.5\textwidth]{search_costs_graph}
\caption{Search costs (expected waiting times to trade) vs transaction tax rate (bp). Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors).}
\end{figure}

As taxes increase, the waiting time between trades increases. The effect of a transaction tax on search costs is the inverse of the results for fill rates: a 50 bp tax roughly doubles search costs.

4.7. Expected Quote Revenue. The equilibrium expected quote revenue per share, $R_Q^e$ is shown in Figure 8. Without market makers, the expected quote revenue decreases from $0.064$ (no tax) to $0.032$ (50 bp tax) — a 50% decrease. When half the arriving traders are market makers, the tax has a greater effect: the expected quote revenue decreases from $0.074$ to $0.027$ — a 64% decrease. Without a tax, the expected quote revenue increases with the proportion of arriving traders being market makers up to about 40%; for 50% of arriving traders being market makers, the expected quote revenue falls slightly. In all of
these cases, however, the expected quote revenue drops faster with taxes as we increase the proportion of market makers. Thus the expected benefit of providing liquidity falls by about half to two-thirds for a 50 bp tax. To the extent that market makers have costs of doing business comparable to expected quote revenues, we could expect market makers to exit the market more than this model suggests.24

![Expected Quote Revenue vs Transaction Tax](image)

**Figure 8.** Expected quote revenues vs transaction tax (bp). Curves show 10% increments of market makers, from none to half of potential traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). Without market makers, the expected quote revenues drop by about half for a 50 bp tax; with half the arriving traders being market makers, the expected quote revenues drop by about two-thirds for a 50 bp tax. This model does not include costs which might imply even greater reductions in expected revenues.

4.8. **Realized Volatility.** The plot of realized volatility in Figure 9 shows that the effects of a transaction tax are not always increasing: there are regimes where a tax reduces the realized volatility.

For markets where 30% or more of the arriving traders are market makers, a transaction tax increases the realized volatility monotonically. For example, a market where half of the arriving traders are market makers has an untaxed realized volatility of about $0.11 and a

24While this model does not allow market makers to exit the market, they can publish quotes which greatly reduce their probability of trading.
realized volatility of about $0.22 for a tax of 50 bp — a doubling of volatility. Furthermore, markets with more market makers have lower realized volatility for taxes up to 50 bp and are more sensitive to a transactions tax; above 50 bp, the volatility is roughly the same for all proportions of market makers. This suggests that at a tax rate of 50 bp almost all market makers have (effectively) exited the market.

However, for markets where 20% or less of the arriving traders are market makers, a transaction tax initially lowers volatility. The decrease is small, but the most extreme decrease is for a market without market makers. Without market makers, the volatility decreases from $0.193 without a tax to $0.186 for a 12 bp tax (a 4% decrease). The initial decrease in volatility is because an increase in taxes lowers fill rates which makes taking a worse price (i.e. a price closer to \( v \)) more likely. However, this is a second-order effect which is dominated by the increase in spreads as taxes increase.

![Figure 9. Volatility (of trade prices) vs transaction tax rate (bp). Curves show 10% increments of market makers, from none to half of potential traders. Market makers have a reservation value offset of 0 while investors have an offset with volatility of 2.5%. More market makers leads to lower volatility yet makes market volatility more sensitive to taxes: a 50 bp tax doubles volatility. Taxes up to about 15 bp decrease volatility for markets without market makers but increase volatility by about one-fifth for a 50 bp tax.](image-url)
4.9. **Gains from Trade.** Since market makers are intermediaries, they are unlikely to take a directional bet. Therefore we might expect that they will trade closer to the mean reservation value and thus for lower expected revenue than most other traders. This is indeed the case: increasing the fraction of market makers lowers the average gains from trade. (One could also view this as market makers reducing the dispersion in beliefs about prices.)

While market makers intermediate for lower returns, they also (in our model) lower volume because they compete for liquidity. Therefore we might expect market makers to lower total gains from trade. Figure 10 shows that to be true; total gains from trade are higher without market makers. For all markets, regardless of the proportion of arriving traders who are market makers, the total gains from trade drop about 60% for a 50 bp tax. Were we to correct the gains for trade by only examining investors, we would see that the gains for trade for investors are higher with market makers.

![Figure 10](image)

**Figure 10.** Total gains from trade vs transaction tax rate (bp) for 100,000 simulated traders. Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). A 50 bp tax decreases the total gains from trade by about 60%. Market makers lower total gains from trade since they trade for small margins.

4.10. **Deadweight Loss.** Since we are considering gains from trade and tax revenue, it makes sense to think about the costs and benefits of the tax. We therefore look at the
deadweight loss: the reduction in gains from trade less tax revenues.\textsuperscript{25} If there were an externality that we could price, we could also consider that.\textsuperscript{26} Figure 11 shows the deadweight loss per order. We look at the loss per order because considering the loss per trade would ignore the damaging effects of reduced volume. The plot shows that for no positive tax level is the deadweight loss negative — implying that the socially optimal tax (in this model economy) is none. However, this ignores the possible benefits of eliminating manipulative or distorting trades. We deal with that issue in Section 4.12.

![Figure 11. Deadweight loss per order vs transaction tax rate (bp) for 100,000 simulated traders. Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). At no tax level is the deadweight loss negative, suggesting that the socially optimal tax is 0 bp.](image)

4.11. Revenue-Optimal Tax Rates. Since one argument for transaction taxes is their revenue-generating capabilities, we consider the maximal government revenue possible in our model. The Laffer curves in Figure 12 show that the maximum government revenue of 14–24 bp per order would be generated at tax rates of 57–69 bp per trade with lower taxes being revenue-optimal for greater proportions of market makers.

\textsuperscript{25}Thanks to Martin Šuster for this idea.

\textsuperscript{26}Volatility could be one such externality; however, the prior analysis of volatility suggests that the tax does little to mitigate volatility.
At the revenue-optimal tax rates, however, other measures reveal a great decrease in market quality. Table 2 shows the various measures of market quality for the different revenue-optimal tax rates. In general, the revenue-optimal tax rates would increase effective spreads (direct trading costs) by 200%–300%, decrease fill rates (volume) by more than half, reduce the reward for providing liquidity by two-thirds, and increase volatility by between one-third and more than double. All of these results suggest conditions that would likely induce trading to move elsewhere as documented in empirical studies. Finally, the revenue raised falls by over 40% between markets with no intermediation and those with half of arriving traders being market makers. This suggests that more developed markets are more sensitive to taxation and may yield lower expected revenue per order.

4.12. Stabilization. One idea mentioned earlier was that a transaction tax might reduce distortions created by “destabilizing speculators.” We investigate this by considering positive-feedback traders as in De Long et al. (1990). In that work, destabilizing speculators impart
TABLE 2. Measures of market quality for revenue-optimal transaction tax rates. Investors have private reserve values for the asset with a mean of $20 and volatility of 2.5%; market makers have reserve values equal to the average of $20. Gains from trade and deadweight loss are displayed in dollars per 1000 orders.

<table>
<thead>
<tr>
<th>% Market Makers</th>
<th>0%</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rev.-optimal Tax (bp)</td>
<td>69</td>
<td>67</td>
<td>64</td>
<td>62</td>
<td>59</td>
<td>57</td>
</tr>
<tr>
<td>Revenue/order (bp)</td>
<td>23</td>
<td>21</td>
<td>19</td>
<td>18</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>Optimal Spread (bp)</td>
<td>305</td>
<td>298</td>
<td>287</td>
<td>280</td>
<td>269</td>
<td>262</td>
</tr>
<tr>
<td>vs untaxed</td>
<td>+377%</td>
<td>+326%</td>
<td>+281%</td>
<td>+249%</td>
<td>+219%</td>
<td>+199%</td>
</tr>
<tr>
<td>Effective Spread (bp)</td>
<td>292</td>
<td>285</td>
<td>274</td>
<td>267</td>
<td>257</td>
<td>250</td>
</tr>
<tr>
<td>vs untaxed</td>
<td>+296%</td>
<td>+271%</td>
<td>+249%</td>
<td>+237%</td>
<td>+223%</td>
<td>+215%</td>
</tr>
<tr>
<td>Fill Rate</td>
<td>34%</td>
<td>32%</td>
<td>30%</td>
<td>28%</td>
<td>27%</td>
<td>24%</td>
</tr>
<tr>
<td>vs untaxed</td>
<td>-55%</td>
<td>-56%</td>
<td>-56%</td>
<td>-57%</td>
<td>-56%</td>
<td>-56%</td>
</tr>
<tr>
<td>E(Quote Revenue) $</td>
<td>0.022</td>
<td>0.022</td>
<td>0.023</td>
<td>0.024</td>
<td>0.025</td>
<td>0.024</td>
</tr>
<tr>
<td>vs untaxed</td>
<td>+34%</td>
<td>-67%</td>
<td>-68%</td>
<td>-68%</td>
<td>-67%</td>
<td>-67%</td>
</tr>
<tr>
<td>Realized Volatility $</td>
<td>0.260</td>
<td>0.259</td>
<td>0.256</td>
<td>0.253</td>
<td>0.247</td>
<td>0.244</td>
</tr>
<tr>
<td>vs untaxed</td>
<td>+34%</td>
<td>+46%</td>
<td>+60%</td>
<td>+79%</td>
<td>+99%</td>
<td>+128%</td>
</tr>
<tr>
<td>Gains from Trade $/k</td>
<td>97.71</td>
<td>91.78</td>
<td>88.01</td>
<td>83.86</td>
<td>77.30</td>
<td>70.15</td>
</tr>
<tr>
<td>vs untaxed</td>
<td>-71%</td>
<td>-71%</td>
<td>-69%</td>
<td>-68%</td>
<td>-66%</td>
<td>-65%</td>
</tr>
<tr>
<td>Deadweight Loss $/k</td>
<td>147.5</td>
<td>136.1</td>
<td>119.8</td>
<td>104.1</td>
<td>88.8</td>
<td>73.6</td>
</tr>
</tbody>
</table>

“positive feedback” to the market; in other words, they mimic the trade that preceded them. This pushes markets further from equilibrium and should yield a loss in allocative efficiency.

In our constant-asset-value setup, this is equivalent to having traders whose reservation value depends in part on the reservation value of the trader preceding them. We structure this as an AR(1) process to maintain covariance stationarity. The dynamics would thus look like this:

(21) \( d_t|d_{t-1} \sim f = \begin{cases} 0 & w.p. \mu \\
N \left( 0, L^2 \left( 1 + \frac{\gamma^2}{1-\gamma^2} \right) \right) & w.p. 1-\mu.
\end{cases} \)

Therefore, the preceding results still hold: Even in the presence of positive-feedback traders, we can expect a tax to widen spreads and increase volatility. Intuitively, this
makes sense: positive-feedback traders take more extreme views than normal (non-positive-feedback) investors; therefore, they are less likely to be dissuaded from trading by a comparatively-small tax. In comparison, normal investors are more likely to be dissuaded by such a tax and, thus, the influence of positive-feedback traders is likely to be proportionally greater with a tax. If we believe positive-feedback traders generate externalities (such as excess volatility), a transactions tax would be welfare-reducing since it does nothing to reduce those externalities and may even exacerbate them.

![Graph showing dispersion of reservation prices for investors who are price takers for 100,000 simulated traders. Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). Note that taxes increase the dispersion of investors who trade — by chasing away investors with less extreme views. The plot for investors who are price makers (i.e. who quote) is the same.]

**Figure 13.** Dispersion of reservation prices for investors who are price takers for 100,000 simulated traders. Curves show 10% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5% (investors). Note that taxes increase the dispersion of investors who trade — by chasing away investors with less extreme views. The plot for investors who are price makers (i.e. who quote) is the same.

4.13. **Asset Prices.** So far, our model has assumed a constant asset price. However, we should consider how a transaction tax might affect asset prices. The difficulty in such analysis is that we cannot just assume some effect on expected returns and an average holding period; Amihud and Mendelson’s (1986) Proposition 1 implies that a wider spread

---

27 Some commenters have likened this to an ostensibly Pigouvian tax on alcohol: A small tax does little to change the drinking habits of alcoholics but does dissuade moderate drinkers.

28 We thank Yakov Amihud for suggesting this subsection.
asset will be held by investors with a longer holding period. Therefore, we consider our findings in light of the numerical example in their Table 1. The effects on asset prices are shown in Figure 14.

\[\text{Figure 14. Relative asset prices based on effective spreads for 100,000 simulated traders. Curves show 10\% increments of market makers, from none to half of arriving traders. Traders have a reservation value offset of 0 (market makers) or with volatility of 2.5\% (investors). Note that taxes reduce asset prices by over 3\% for a 10 bp tax, about 8\% for a 25 bp tax, and about 10\% for a 50 bp tax.}\]

Obviously, there are more than four investment horizons for investors, so the plot in Figure 14 might be more curved in reality; however, the order of magnitude for the effect on asset prices would not change much (and might be worse for a 10 bp tax). While coupling two unrelated models is an imperfect exercise, the results suggest we can expect asset prices to decline by over 3\% for a 10 bp tax, about 8\% for a 25 bp tax, and about 10\% for a 50 bp tax. If we were to include the effects of higher volatility, the effect on asset prices would be even greater. Thus we believe that a transaction tax would have a strong effect on asset prices and that this reduction is not muted for “small” levels of a tax. Finally, we should note that we have only assumed that the tax affects the spread. Since we have shown that

\[\text{Because we do not have a zero-spread instrument, we use the zero-tax spread as the baseline for determining the reduction in asset prices.}\]
the tax also affects volatility, we should expect asset price declines greater than those in Figure 14.

5. Postscript

Since the start of this research, some countries have considered or implemented a financial transactions tax. European politicians have been the most vocal proponents of such a tax: European Commission (2013) proposes that a number of EU member states would implement a 10 bp tax on non-derivative financial transactions and a 1 bp tax on derivative transactions. France and Italy have already acted: On 14 April 2012, French lawmakers proposed a 20 bp tax on equities with a market cap of €1B or more; that passed on 1 August 2012 and went into effect on 30 November 2012. Italy proposed a 10 bp tax on equities with a market cap of €500M or more on 22 November 2012; that passed on 29 December 2012 and went into effect on 1 March 2013.

While the effects of these taxes are still being studied, we find some early evidence in support of our model. Capelle-Blancard and Havrylchyk (2014) find volumes decrease by about 20% for taxed vs. untaxed French equities — as well as increased volumes in similar equities in untaxed markets, suggesting some portfolio substitutions. However, they are unable to find statistically significant differences in volatility or price impact. This may be due a data issue: their “pre-tax” period of February–July 2012 is nearly bisected by the date at which the tax was introduced as legislation. Rühl and Stein (2014) examine taxed Italian equities and compare them to similar, untaxed stocks; they find that volatility increases about 18% and spreads increase by about 1.7% post-tax. They also find insignificant evidence of reduced volumes in taxed stocks.

For evidence of the effect on asset prices, we collected data from December 2011–November 2014 for seven European equity indices: CAC 40, FTSE MIB, DAX, FTSE 100, AEX, OMX, and SMI. We then examined excess returns for the French CAC 40 and Italian FTSE MIB beyond returns averaged across the other five large European equity indices. Figure 15 shows these excess returns for the CAC 40 and MIB for a one-year window centered on the tax.
announcement date. Since the introduction of the transactions tax as legislation, the CAC 40 and MIB have been discounted an average of 4% and 3.4% versus the average of the other European equity indices.

![Cumulated returns of the CAC 40 and MIB versus an average of DAX, FTSE, AEX, SMI, and OMX. Times shown are one-year windows around the proposal of the tax with the proposal date denoted by a vertical line.](image)

**Figure 15.** Cumulated returns of the CAC 40 and MIB versus an average of DAX, FTSE, AEX, SMI, and OMX. Times shown are one-year windows around the proposal of the tax with the proposal date denoted by a vertical line.

We compare these results with what our model suggests in Table 3, assuming France and Italy have high levels of financial intermediation (i.e. 50% of trades involving an intermediary). In general, our model seems to be closer to observed effects for volume, volatilities, and declines in asset values while not as accurate for quoted spreads. Further research on a longer dataset is needed to understand these effects, differentiate the effects of a transactions tax from a quote tax (also instituted in France), and discern the amount of tax avoidance.

### 6. Conclusion

Following the recent financial crisis, regulators have proposed a securities transaction tax in the hopes of reducing price volatility encouraging long-term investing, raising large amounts of revenue from a very small tax, and pushing harmful speculators out of the market. Opponents, however, have argued that a transaction tax will reduce liquidity and increase trading costs making trading too expensive for some investors. They have also said a tax such as the broad-based one proposed by DeFazio will be difficult to implement (especially across asset classes); will distort the market by reducing market efficiency; and, will push
traders to other venues or countries. Recent evidence from new taxes in Europe seems to confirm some of these concerns.

However, policy makers prefer not to experiment with their markets and interpreting empirical studies with respect to proposed taxes is often unclear. We developed this model to guide policy makers and to help academics understand how different aspects of market quality may be related or affected by a tax. The resulting sequential trader model is very clean in its assumptions — a contribution to the market microstructure literature in its own right. However, the model also has much to say about the effects of intermediation in thin/fast markets as well as the effects of financial transaction taxes.

The model suggests that a tax increases the effects of destabilizing speculators; it does not chase them out of the market. We also find that optimal and effective spreads widen; volatility may slightly decrease (while other metrics get much worse) or, more often, increases greatly; both the benefits of providing liquidity and gains from trade decrease; and, asset prices decline. All of these effects are strong enough to be highly economically significant. The suggested effects on liquidity and trading costs would hurt investors while the effects on volatility and asset prices would reduce risk taking and capital raising in primary markets. Thus a tax could hurt job and wealth creation. Furthermore, our analysis suggests that the

<table>
<thead>
<tr>
<th></th>
<th>France: 20 bp tax Model</th>
<th>Estimated</th>
<th>Italy: 10 bp tax Model</th>
<th>Estimated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume</td>
<td>-13%</td>
<td>-20%</td>
<td>-9%</td>
<td>—</td>
</tr>
<tr>
<td>Volatility</td>
<td>+19%</td>
<td>—</td>
<td>+10%</td>
<td>+18%</td>
</tr>
<tr>
<td>Spread</td>
<td>+30%</td>
<td>—</td>
<td>+13%</td>
<td>+1.7%</td>
</tr>
<tr>
<td>Asset Prices</td>
<td>-3.5%</td>
<td>-4.0%</td>
<td>-2.1%</td>
<td>-3.4%</td>
</tr>
</tbody>
</table>

Table 3. Effects of financial transaction taxes enacted in France and Italy in 2012. Model results are those predicted by our model for a tax of half the quoted size (since France and Italy make only the buyer pay). Empirical results for volumes, volatilities, and spreads come from Capelle-Blancard and Havrylchyk (2014) (France) and Rühl and Stein (2014) (Italy). Fields with a “—” indicate insignificant findings. Empirical results for asset prices are from our own analysis.
revenue-optimal tax would be very high, between 55 bp and 70 bp. These rates would be very harmful to market quality.

Increasing the fraction of market makers causes markets to be more sensitive to transaction taxes. Under a tax, volatility increases much faster than in a market lacking intermediaries. Volume also drops by a greater fraction as do the benefits of providing liquidity and gains from trade. Thus markets with high levels of intermediation may respond differently to a tax than markets with little or no intermediation. This is crucial for policy makers because it suggests that the findings of empirical studies may not be applicable to markets with differing levels of intermediation.

Finally, there are many issues which have been left for further work. For example, we do not address the effects of quote taxes (recently implemented in France). However, we believe that prior theory will help us conduct a similar study of those effects. Taken with the work here, that would then allow us to examine the French policy to tease apart the effects of the two taxes. We also suspect this would help us determine the fraction of people somehow escaping the tax as well as the propensity of traders to leave for other (less taxed) markets.

APPENDIX A. GENERAL CHARACTERIZING PROOFS


Proof of Proposition 1. In Table 1, Rick takes Ilsa’s prices if \( d_{t+1} < -R_Q^*|d_{t+1} - \delta - 2\tau \) or \( d_{t+1} > R_Q^*|d_{t+1} + \beta + 2\tau \). Suppose both could be satisfied. Then we would have that:

\[
R_Q^*|d_{t+1} + \beta + 2\tau < -R_Q^*|d_{t+1} - \delta - 2\tau, \text{ which implies}
\]

\[
(22) \quad \beta + \delta < -2R_Q^*|d_{t+1} - 4\tau.
\]

Since \( R_Q^*|d_{t+1} \) and \( \tau \) are non-negative, this implies a negative bid-ask spread — and we have a contradiction. \( \square \)

A.2. Characterizing the Bid-Ask Spread.
Proof of Proposition 2. We prove the case for \( d_t > 0 \); the case for \( d_t < 0 \) proceeds similarly. Suppose \( \delta^* \geq \beta^* \). Let Ilsa’s optimal conditional expected revenue (with the conditioning on \( d_t \) suppressed for space) be given by:

\[
R^*_Q = \rho F(-R^*_Q - \delta^* - 2\tau)(\delta^* + d_t - 2\tau) + \rho F(-R^*_Q - \beta^* - 2\tau)(\beta^* - d_t - 2\tau).
\]

We define a sub-optimal revenue \( R'_Q \) by switching \( \delta^* \) and \( \beta^* \):

\[
R'_Q = \rho F(-R^*_Q - \beta^* - 2\tau)(\beta^* + d_t - 2\tau) + \rho F(-R^*_Q - \delta^* - 2\tau)(\delta^* - d_t - 2\tau).
\]

Since \( R^*_Q \) is optimal, we have that \( R^*_Q - R'_Q > 0 \). However,

\[
R_Q - R'_Q = \rho 2d_t \left( F(-R^*_Q - \delta^* - 2\tau) - F(-R^*_Q - \beta^* - 2\tau) \right) < 0
\]

is negative by the properties of a cdf. This implies \( R^*_Q < R'_Q \) and that \( \delta^* \) and \( \beta^* \) are not optimal — a contradiction.

Finally, if \( d_t = 0 \), the optimal revenue is given by:

\[
R^*_Q = \rho F(-R^*_Q - \delta^* - 2\tau)(\delta^* - 2\tau) + \rho F(-R^*_Q - \beta^* - 2\tau)(\beta^* - 2\tau).
\]

These two summands must have the same maximizer, so \( \delta^* = \beta^* \).

\[
\square
\]

Proof of Proposition 3. Define \( H(x) = F(G(x))/f(G(x)) \), then we can rewrite the equilibrium \( \delta^* \) and \( \beta^* \) as,

\[
\delta^* = H(\delta^*) - d_t + 2\tau, \quad \beta^* = H(\beta^*) + d_t + 2\tau, \quad \text{and thus,}
\]

\[
\delta^* + \beta^* = H(\delta^*) + H(\beta^*) + 4\tau \geq 4\tau.
\]

Since we assumed \( \tau/\text{share} \) is collected at position entry and exit (a total of \( 2\tau \)), the bid-ask spread \( \delta^* + \beta^* \) is more than twice the tax collected.

\[
\square
\]

A.3. Quoting Benefit is Positive.
Proof of Proposition 4. Rearranging the terms for \( R^*_Q|d_t \), we have

\[
R^*_Q|d_t = \rho F(G(\delta^*)) (\delta^* + d_t - 2\tau) + \rho F(G(\beta^*)) (\beta^* - d_t - 2\tau)
\]

\[
= \rho [(F(G(\delta^*)) - F(G(\beta^*))])d_t + \rho F(G(\delta^*)) (\delta^* - 2\tau) + \rho F(G(\beta^*)) (\beta^* - 2\tau). \tag{30}
\]

We prove the case for \( d_t > 0 \); the case for \( d_t < 0 \) proceeds similarly. If \( d_t > 0 \), we have \( \beta^* > \delta^* \) from Proposition 2. Therefore, we have \( G(\delta^*) > G(\beta^*) \) and \( F(G(\delta^*)) > F(G(\beta^*)) \), as \( G(\cdot) \) is a decreasing function and \( F(\cdot) \) is an increasing function by definition. It follows that,

\[
F(G(\delta^*)) (\delta^* - 2\tau) + F(G(\beta^*)) (\beta^* - 2\tau) > F(G(\beta^*)) (\delta^* - 2\tau) + F(G(\beta^*)) (\beta^* - 2\tau) \tag{32}
\]

\[
> F(G(\beta^*)) (\delta^* - 2\tau) + F(G(\beta^*)) (\beta^* - 2\tau) \tag{33}
\]

\[
= F(G(\beta^*)) (\delta^* + \beta^* - 4\tau) > 0 \tag{34}
\]

where the last inequality follows from Proposition 3. \( \square \)


Proof of Proposition 5. Rewrite the expected quote profit \( R_Q|d_t \) as,

\[
R_Q|d_t \equiv \rho w(\delta, \beta) = \rho w_1(\delta) + \rho w_2(\beta) \quad \text{where}
\]

\[
w_1(\delta) = F(G(\delta))(\delta + d_t - 2\tau), \quad \text{and}
\]

\[
w_2(\beta) = F(G(\beta))(\beta - d_t - 2\tau). \tag{37}
\]

Holding \( R^*_Q, \tau, \) and \( d_t \) fixed, we have that

\[
\lim_{\delta \to -\infty} F(G(\delta))(\delta + d_t - 2\tau) = -\infty; \quad \text{and,}
\]

\[
\lim_{\delta \to -\infty} F(G(\delta))(\delta + d_t - 2\tau) = \lim_{\delta \to -\infty} F(-\delta)\delta = -\lim_{\delta \to -\infty} F(\delta)\delta = 0 \tag{39}
\]
with the last equality justified by finite variance: \( \text{Var}(\delta) < \infty \) implies the cdf converges to 0 faster than \( \delta \).

We then note that \( \delta + d_t - 2\tau > 0 \) implies \( \delta > 2\tau - d_t \), in which case:

\[
(40) \quad w_1(\delta) = F(G(\delta))(\delta + d_t - 2\tau) > 0.
\]

Since \( R_Q^c \) and \( \tau \) are bounded, we have that:

\[
(41) \quad \max_\delta w_1(\delta) = \max_\delta F(G(\delta))(\delta + d_t - 2\tau) = F(G(\delta_M))(\delta_M + d_t - 2\tau)
\]

for some \( \delta_M > 2\tau - d_t \). We can also show likewise for \( w_2(\beta) \): That there exists a maximizing \( \beta_M > 2\tau + d_t \). Therefore, we have that

\[
(42) \quad \max_{\delta, \beta}(w_1(\delta) + w_2(\beta)) = w_1(\delta_M) + w_2(\beta_M) > 0.
\]

Since we attain a bounded maximum at some \( \delta_M, \beta_M \) for all \( d_t \) over the support of the distribution, there is an expected best strategy and thus a Markov Perfect equilibrium exists. \( \square \)


Proof of Proposition 6. Continuing with the notation in the preceding proof, we find the optimal \( \delta_M \) and \( \beta_M \) by differentiating \( w_1 \) and \( w_2 \). This gives us our first-order conditions:

\[
(43) \quad \frac{\partial w_1}{\partial \delta}\bigg|_{\delta=\delta_M} = -f(G(\delta_M))(\delta_M + d_t - 2\tau) + F(G(\delta_M)) = 0
\]

\[
(44) \quad \frac{\partial w_2}{\partial \beta}\bigg|_{\beta=\beta_M} = -f(G(\beta_M))(\beta_M - d_t - 2\tau) + F(G(\beta_M)) = 0.
\]

The equilibrium expected quote revenue, \( R_Q^c \), is then the fixed point satisfying the following equilibrium condition:

\[
(45) \quad R_Q^c = \psi(R_Q^c) \quad \text{where}
\]

\[
(46) \quad \psi(R_Q^c) \equiv E[E(R_Q^c|d_t)] = \int_{-\infty}^{\infty} (w_1(\delta_M(R_Q^c)) + w_2(\beta_M(R_Q^c)))dF(d_t).
\]
Since $F$ is continuous, the pdf $f$ is dominated by some function and Fatou’s Lemma allows us to interchange integration and limit operations. This lets us differentiate under the integral sign, yielding

\[ \frac{\partial \psi}{\partial R} \bigg|_{Q} = \int_{-\infty}^{\infty} \left( \frac{\partial w_1(\delta_M)}{\partial R} + \frac{\partial w_2(\beta_M)}{\partial R} \right) dF(d_t) \]

for all $R^o_Q \in [0, \infty)$.

Differentiating $w_1$ and $w_2$ with respect to $R^o_Q$, we get

\[ \frac{\partial w_1(\delta_M)}{\partial R^o_Q} = (F(G(\delta_M)) - f(G(\delta_M))(\delta_M + d_t - 2\tau)) \frac{\partial \delta_M}{\partial R^o_Q} \]

\[ - f(G(\delta_M))(\delta_M + d_t - 2\tau), \]

and

\[ \frac{\partial w_2(\beta_M)}{\partial R^o_Q} = (F(G(\beta_M)) - f(G(\beta_M))(\beta_M - d_t - 2\tau)) \frac{\partial \beta_M}{\partial R^o_Q} \]

\[ - f(G(\beta_M))(\beta_M - d_t - 2\tau). \]

The first-order conditions on $w_1$ and $w_2$ then reduce these to

\[ \frac{\partial w_1(\delta_M)}{\partial R^o_Q} = -f(G(\delta_M))(\delta_M + d_t - 2\tau) < 0, \]

\[ \frac{\partial w_2(\beta_M)}{\partial R^o_Q} = -f(G(\beta_M))(\beta_M - d_t - 2\tau) < 0. \]

Combining these and invoking the continuity of $F$, we then have that $R^o_Q(R_Q|d_t)$ is a strictly decreasing continuous function. Therefore, for all $\rho \in (0, 1]$, there must be a unique $R^o_Q$ such that $R_Q(R^o_Q|d_t) = R^o_Q/\rho$. Thus the Markov Perfect equilibrium is unique. \[\square\]

**Appendix B. Uniform Investors Without Intermediation.**

*Proof of Corollary 1.* We compute the unconditional expectation of $(R^o_Q|d_t)/\rho$:

\[ \frac{R^o_Q}{\rho} = E[E(R^o_Q|d_t)/\rho] \]

\[ = \int_{-L+R^o_T+4\tau}^{L} (\delta^* + d_t - 2\tau)^2 dd_t + \int_{-L-R^o_T-4\tau}^{-L} (\beta^* + d_t - 2\tau)^2 dd_t \]

\[ = \frac{\int_{-L+R^o_T+4\tau}^{L} (\delta^* + d_t - 2\tau)^2 dd_t + \int_{-L-R^o_T-4\tau}^{-L} (\beta^* + d_t - 2\tau)^2 dd_t}{4L^2} \]
TRANSACTION TAXES IN A PRICE MAKER/TAKER MARKET

\[
= (d_t - R_Q^\circ + L - 4\tau)^3/(3 \cdot 16L^2)|_{L+R_Q^\circ+4\tau}^L + (d_t + R_Q^\circ - L + 4\tau)^3/(3 \cdot 16L^2)|_{L-R_Q^\circ-4\tau}^L
\]

(54)

\[
= (2L - R_Q^\circ - 4\tau)^3/(24L^2) \implies R_Q^\circ = \frac{\rho}{24L^2}(2L - R_Q^\circ - 4\tau)^3.
\]

(55)

Since the cubic term has a triple root, the above equation has a unique solution for \(R_Q^\circ\).

Differentiating with respect to tax \(\tau\) or continuation probability \(\rho\), we get

\[
\frac{\partial R_Q^\circ}{\partial \tau} = -\frac{\rho(2L - R_Q^\circ - 4\tau)^2}{2L^2 + \frac{\rho}{4}(2L - R_Q^\circ - 4\tau)^2} < 0,
\]

(56)

\[
\frac{\partial R_Q^\circ}{\partial \rho} = \frac{(2L - R_Q^\circ - 4\tau)^3}{24L^2 + 3\rho(2L - R_Q^\circ - 4\tau)^2} > 0.
\]

(57)

This implies that

\[
\frac{\partial \delta^*}{\partial \tau} > 0, \quad \frac{\partial \beta^*}{\partial \tau} > 0, \quad \frac{\partial \delta^*}{\partial \rho} < 0, \quad \frac{\partial \beta^*}{\partial \rho} < 0.
\]

(58)

\[\square\]

Proof of Proposition 8. Proceeding in order from the proposition:

(1) Effective Spread

\[
E(Ilsa’s \ sell \ price | Ilsa \ sells) - E(Ilsa’s \ buy \ price | Ilsa \ buys)
\]

(59)

\[
= E(E(\beta^* | Ilsa \ sells)) + E(E(\delta^* | Ilsa \ buys))
\]

(60)

\[
= E\left(\frac{1}{2}(L - R_Q^\circ + d_t) | \text{Ilsa sells}\right) + E\left(\frac{1}{2}(L - R_Q^\circ - d_t) | \text{Ilsa buys}\right)
\]

(61)

\[
= L - R_Q^\circ + E(d_t | \text{Ilsa sells}) - E(d_t | \text{Ilsa buys}).
\]

(62)

From Table 1, Ilsa sells if \(d_{t+1} > R_Q^\circ(d_{t+1}) + \beta^*(d_t) + 2\tau\) and buys if \(d_{t+1} < -R_Q^\circ(d_{t+1}) - \delta^*(d_t) - 2\tau\).

(63)

\[
L - R_Q^\circ + E(d_t | \text{Ilsa sells}) - E(d_t | \text{Ilsa buys})
\]

(64)

\[
= L - R_Q^\circ + E(E(d_t | d_{t+1} > R_Q^\circ(d_{t+1}) + \frac{L - R_Q^\circ + d_t}{2} + 2\tau))
\]
\begin{align*}
+ E(E(d_t|d_{t+1} < -R_Q^*(d_{t+1}) - \frac{L - R_Q^* - d_t}{2} - 2\tau)) \\
= L - R_Q^* + \frac{1}{2L} \left[ \frac{(R_Q^* + 4\tau)^2}{8} - \frac{L^2}{2} \right]
\end{align*}

(2) Fill Rate

\begin{align*}
P(buy) = 1 - F(R_Q^* + \beta^* + 2\tau) &= \frac{L - R_Q^* - \beta^* - 2\tau}{2L} \\
&= \frac{1}{2} \left( L - R_Q^* \right) - \frac{1}{2} d_t - 2\tau \\
P(sell) = F(-R_Q^* - \delta^* - 2\tau) &= \frac{L - R_Q^* - \delta^* - 2\tau}{2L} \\
&= \frac{1}{2} \left( L - R_Q^* \right) + \frac{1}{2} d_t - 2\tau.
\end{align*}

P(buy) is greater or equal than zero when \( d_t \) lies in the range of \(-L < d_t < L - R_Q^* - 4\tau\) and P(sell) is not less than zero only when \( R_Q^* + 4\tau - L < d_t < L \). Therefore

\begin{align*}
E[P(buy)] &= \frac{\left[ 2L - (R_Q^* + 4\tau) \right]^2}{16L^2}, \text{ and} \\
E[P(sell)] &= \frac{\left[ 2L - (R_Q^* + 4\tau) \right]^2}{16L^2}.
\end{align*}

(3) Expected Search Cost

Let us denote \( P(\text{Not Filled}) \) by \( q \). Then the expected search cost can be written as \( C_S = q + 2q^2 + 3q^3 + \cdots \). Multiplying both sides of the equation by \( q \), we have \( qC_S = q^2 + 2q^3 + \cdots \).

Manipulating these, we get \((1 - q)C_S = q + q^2 + q^3 + \cdots = \frac{q}{1-q} \). Therefore, \( C_S = \frac{q}{(1-q)^2} \). Substituting \( q = 1 - \text{Fill Rate} \) completes our calculation.

(4) Asset Price

Let \( p \) denote Ilsa’s buy price when Rick sells, and her sell price when Rick buys. If Rick decides to quote, a transaction price would not exist in that period — so we cannot infer the asset price and exclude that event.

\begin{align*}
E[p] = E[E[p|d_t]]
\end{align*}
(74) \[ E[E[p|buy]P(buy|d_t) + E[p|sell]P(sell|d_t)]. \]

Conditional on the event “buy”, \( p = v - \delta^* \). Then we have

(75) \[ E[p|buy] = E[v - \delta^*|buy] = v - E[\delta^*|buy] \]

(76) \[ = v - \frac{1}{2}(L - R_Q) + \frac{1}{2}E[d_t|buy] \]

From the proof of the effective spread, we have

(77) \[ E[p|buy] = v - \frac{1}{2}(L - R_Q) + \left[ \frac{L^2}{2} - \frac{(R_Q + 4\tau)^2}{8} \right] \frac{1}{4L}. \]

Similarly, conditional on the event “sell”, \( p = v + \beta^* \) and we have

(78) \[ E[p|sell] = v + \frac{1}{2}(L - R_Q) + \left[ \frac{(R_Q + 4\tau)^2}{8} - \frac{L^2}{2} \right] \frac{1}{4L}. \]

\[ E[P(buy|d_t)] = E[P(buy)] \] and \( E[P(sell|d_t)] = E[P(sell)] \) are found in the fill rate proof. Inserting those into the expression for \( E[p] \) completes the proof.

\[ \square \]

**APPENDIX C. UNIFORM INVESTORS WITH INTERMEDIATION**

Using the definitions in equations (13)–(16), we first prove a number of lemmas necessary for the main result.

**Proof of Lemma 1.** By the nature of CDFs and indicator functions, \( V_1 \) and \( V_2 \) are non-negative for \( \delta > 2\tau - d_t, \beta > 2\tau + d_t \).

“ \( \iff \) ” Since \( d_t > R_Q^o + 4\tau - L \), we have \( 2\tau - d_t < L - R_Q^o - 2\tau \). For any \( \delta \in (2\tau - d_t, L - R_Q^o - 2\tau) \), we have: \( \delta + d_t - 2\tau > 0 \) and \( G(\delta) = -R_Q^o - \delta - 2\tau > -L \). Therefore, we know that \( F(G(\delta)) > 0 \) and \( (1 - \mu)F(G(\delta)) + \mu \mathbb{I}(G(\delta)) > 0 \). Thus \( V_1(\delta) > 0 \) and so \( \max_\delta V_1(\delta) > 0 \).

“ \( \implies \) ” \( \max_\delta V_1(\delta) > 0 \) implies that \( \exists \delta \) s.t. \( V_1(\delta) > 0 \). Hence \( G(\delta) > -L \) and \( \delta + d_t - 2\tau > 0 \). Therefore, \( 2\tau - d_t < \delta < L - R_Q^o - 2\tau \) which implies that \( d_t > R_Q^o + 4\tau - L \).
The case for $\beta$ is similar.

\begin{align*}
\frac{\partial V_1(\delta)}{\partial \delta} &= \begin{cases} 
(1 - \mu) + \mu = 1 & G(\delta) > L \\
\frac{1 + \mu}{2} - \frac{(1 - \mu)(R_Q^0 + 2\delta + d_t)}{2L} & 0 \leq G(\delta) \leq L \\
\frac{1 - \mu}{2} - \frac{(1 - \mu)(R_Q^0 + 2\delta + d_t)}{2L} & -L \leq G(\delta) \leq 0 \\
0 & G(\delta) < -L,
\end{cases} \\
\frac{\partial V_2(\beta)}{\partial \beta} &= \begin{cases} 
(1 - \mu) + \mu = 1 & G(\beta) > L \\
\frac{1 + \mu}{2} - \frac{(1 - \mu)(R_Q^0 + 2\beta + d_t)}{2L} & 0 \leq G(\beta) \leq L \\
\frac{1 - \mu}{2} - \frac{(1 - \mu)(R_Q^0 + 2\beta + d_t)}{2L} & -L \leq G(\beta) \leq 0 \\
0 & G(\beta) < -L.
\end{cases}
\end{align*}

Proof of Lemma 2. By inspection of equations (79) and (80).

Corollary 5. For $d_t \in (R_Q^0 + 4\tau - L, L]$, if $\delta \leq -R_Q^0 - 2\tau$, then $\frac{\partial V_1(\delta)}{\partial \delta} > 0$. For $d_t \in [-L, L - R_Q^0 - 4\tau)$, if $\beta \leq -R_Q^0 - 2\tau$, then $\frac{\partial V_2(\beta)}{\partial \beta} > 0$.

Proof. Let $\delta_0 = -R_Q^0 - 2\tau$ and $\beta_0 = -R_Q^0 - 2\tau$. Since the second order derivatives are non-positive, for $\delta \leq \delta_0$:

\begin{align*}
\frac{\partial V_1(\delta)}{\partial \delta} \geq \left. \frac{\partial V_1(\delta)}{\partial \delta} \right|_{\delta = \delta_0} &= \frac{1 - \mu}{2L}(-L - R_Q^0 - 2\delta_0 - d_t + \frac{2L}{1 - \mu}) \\
&= \frac{1 - \mu}{2L}(R_Q^0 + 4\tau - L - d_t) + 1 > \frac{1 - \mu}{2L}(-2L) + 1 = \mu > 0.
\end{align*}

Similarly, for $\beta \leq \beta_0$ $\frac{\partial V_2(\beta)}{\partial \beta} \geq \left. \frac{\partial V_2(\beta)}{\partial \beta} \right|_{\beta = \beta_0} > 0$.

Corollary 6. There exists a unique $\delta_M > \delta_0$ such that:

\begin{align*}
\frac{\partial V_1(\delta)}{\partial \delta} \bigg|_{\delta = \delta_M} = 0 \text{ where } \delta_M = \frac{1}{2}(L - R_Q^0 - d_t)
\end{align*}
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for \( d_t \in (R^o_Q + 4\tau - L, L) \). There also exists a unique \( \beta_M > \beta_0 \) such that:

\[
\frac{\partial V_2(\beta)}{\partial \beta} \bigg|_{\beta = \beta_M} = 0 \quad \text{where} \quad \beta_M = \frac{1}{2}(L - R^o_Q - d_t)
\]

for \( d_t \in [-L, L - R^o_Q - 4\tau) \).

We then have \( V_1(\delta_M) \) defined on \( R^o_Q + 4\tau - L < d_t \leq L \):

\[
V_1(\delta_M) = (1 - \mu)F(G(\delta_M))(\delta_M + d_t - 2\tau)
\]

\[
= \frac{1 - \mu}{8L}(L - R^o_Q + d_t - 4\tau)^2 > 0,
\]

and \( V_2(\beta_M) \) defined on \( -L \leq d_t < L - R^o_Q - 4\tau \):

\[
V_2(\beta_M) = (1 - \mu)F(G(\beta_M))(\beta_M - d_t - 2\tau)
\]

\[
= \frac{1 - \mu}{8L}(L - R^o_Q - d_t - 4\tau)^2 > 0.
\]

**Lemma 3.** \( V_1(\delta_0) > 0 \) iff \( 4\tau < L, R^o_Q < L - 4\tau \) and \( R^o_Q + 4\tau < d_t \leq L \).

**Proof.** We have \( V_1(\delta_0) = \frac{1 + \mu}{2}(d_t - R^o_Q - 4\tau) > 0 \) iff \( d_t > R^o_Q + 4\tau \). Since \( -L \leq d_t \leq L \), the conclusion follows. \( \square \)

By similar proof, we have the following:

**Lemma 4.** \( V_2(\beta_0) > 0 \) iff \( 4\tau < L, R^o_Q < L - 4\tau \) and \( -L \leq d_t < R^o_Q + 4\tau \).

**Lemma 5.** If \( 0 \leq \tau < \frac{\mu L}{\sqrt{2\mu + 2\mu^2 + 2\mu}} \) and \( R^o_Q \leq \frac{4\mu L}{\sqrt{2\mu + 2\mu^2 + 2\mu}} - 4\tau \), then:

1. For any \( d_t \in (x_1 + R^o_Q + 4\tau, L] \), max\( \delta \) \( V_1(\delta) = V_1(\delta_0) \); and,

2. For any \( d_t \in [-L, -x_1 + R^o_Q - 4\tau) \), max\( \beta \) \( V_2(\beta) = V_2(\beta_0) \)

where \( x_1 = \frac{(1 - \mu)L}{1 + 3\mu + 2\sqrt{2\mu + 2\mu^2}} \).

**Proof.**

\[
V_1(\delta_0) > V_1(\delta_M) \iff \quad (89)
\]

\[
\frac{1 + \mu}{2}(-R^o_Q - 4\tau + d_t) - \frac{1 - \mu}{8L}(L - R^o_Q - 4\tau + d_t)^2 > 0,
\]

(90)
$$V_2(\beta_0) > V_2(\beta_M) \iff$$

$$\frac{1 + \mu}{2}(-R_Q - 4\tau - d_t) - \frac{1 - \mu}{8L}(L - R_Q - 4\tau - d_t)^2 > 0.$$

For $d_t \in (R_Q + 4\tau, L]$, let $x = d_t - R_Q - 4\tau$. Then equation (90) $\iff$

$$\frac{1 + \mu}{2}x - \frac{1 - \mu}{8L}(L + x)^2 > 0$$

$$\iff (1 - \mu)x^2 - 2L(1 + 3\mu)x + (1 - \mu)L^2 < 0.$$

By the quadratic formula, we define:

$$x_1 = \frac{(1 - \mu)L}{1 + 3\mu + 2\sqrt{2\mu + 2\mu^2}}, \quad x_2 = \frac{(1 + 3\mu + 2\sqrt{2\mu + 2\mu^2})L}{(1 - \mu)}.$$

Equation (94) holds iff $x_1 < x < x_2$ and, in that case, $x_1 < L < x_2$. (An aside: If $\mu = 0$, $x_1 = x_2 = x = L$ and equation (90) cannot hold.)

Since $d_t \leq L$, $x \leq L - R_Q - 4\tau$; if $x_1 > L - R_Q - 4\tau$, then $d_t > L$ which is not possible — so equation (90) cannot be satisfied. Thus we require

$$L - R_Q - 4\tau \geq x_1 \iff L - x_1 \geq R_Q + 4\tau$$

$$\iff \frac{2L}{1 - \mu}(\sqrt{2\mu + 2\mu^2} - 2\mu) \geq R_Q + 4\tau$$

$$\iff R_Q + 4\tau \leq \frac{4\mu L}{\sqrt{2\mu + 2\mu^2} + 2\mu}.$$

Similarly, for equation (92) to hold, we require that

$$R_Q + 4\tau \leq \frac{4\mu L}{\sqrt{2\mu + 2\mu^2} + 2\mu}.$$
Proof of Proposition 9. First consider the case \( \frac{\mu L}{\sqrt{2\mu + 2\mu^2 + 2\mu}} \leq 2\tau < L \). Assume that \( R^o_Q \in (0, 2L - 4\tau) \). From Lemma 1,

\[
\frac{E[E(R_Q^*|d_t)]}{\rho} = \frac{1 - \mu}{48L^2} (d_t + R^o_Q + 4\tau - L)^3 \bigg|_{-L}^{L - R^o_Q - 4\tau} \\
+ \frac{1 - \mu}{48L^2} (d_t - R^o_Q - 4\tau + L)^3 \bigg|_{R^o_Q + 4\tau - L}^{L} \\
= \frac{1 - \mu}{24L^2} (2L - R^o_Q - 4\tau)^3.
\]

(99)

Since this is a decreasing cubic with a triple root at \( R^o_Q = 2L - 4\tau \), it decreases to 0 on \( R^o_Q \in (0, 2L - 4\tau) \), implying there is a unique equilibrium.

Now consider the case \( 0 \leq \tau < \frac{\mu L}{\sqrt{2\mu + 2\mu^2 + 2\mu}} \). Assume \( R^o_Q \in (0, L + \frac{2L}{1+\mu} - 4\tau) \) and recall \( x_1 = \frac{1+3\mu - \sqrt{2\mu + 2\mu^2 + 2\mu}}{1-\mu} L \). From Lemmas 1 and 5,

\[
\frac{E[E(R_Q^*|d_t)]}{\rho} = \frac{1 + \mu}{8L} (d_t + R^o_Q + 4\tau)^2 \bigg|_{-L}^{-x_1 - R^o_Q - 4\tau} \\
+ \frac{1 - \mu}{48L^2} (d_t + R^o_Q + 4\tau - L)^3 \bigg|_{-x_1 - R^o_Q - 4\tau}^{L - R^o_Q - 4\tau} \\
+ \frac{1 - \mu}{48L^2} (d_t + L - R^o_Q - 4\tau)^3 \bigg|_{R^o_Q + 4\tau - L}^{x_1 + R^o_Q + 4\tau} \\
+ \frac{1 + \mu}{8L} (d_t - R^o_Q - 4\tau)^2 \bigg|_{x_1 + R^o_Q + 4\tau}^{L} \\
= \frac{6(1 + \mu) L((L - R^o_Q - 4\tau)^2 - x^2_1) + (1 - \mu)(L + x_1)^3}{24L^2} \\
= \frac{(1 + \mu)(3(L - R^o_Q - 4\tau)^2 - 3x^2_1 + 2x_1(L + x_1))}{12L} \\
= \frac{(1 + \mu)(3(L - R^o_Q - 4\tau)^2 + 2x_1L - x^2_1)}{12L} > 0.
\]

(100)

(101)

(102)

(103)

(104)

where equation (93) equalling zero for \( x = x_1 \) yields the penultimate step and \( x_1 \leq L \) yields the last step. Note that

\[
\frac{E[E(R_Q^*|d_t)]}{\rho}|_{R^o_Q=0} = \frac{1 + \mu}{12L} (3(L - 4\tau)^2 + x_1(2L - x_1)) > 0
\]

(105)
since \(0 < x_1 < L\) and \(4\tau < L\). Now consider the difference \(\Delta\) between \(E[E(R_Q^*|d_t)]\) and \(R_Q^*\) to see if a fixed point exists:

\[
\Delta(R_Q^*) = \frac{E[E(R_Q^*|d_t)]}{\rho}|_{R_Q^*} - \frac{R_Q^*}{\rho};
\]

(106)

\[
\frac{\partial \Delta(R_Q^*)}{\partial R_Q^*} = \frac{1 + \mu}{12L} 6(L - R_Q^* - 4\tau)(-1) - 1 = 0
\]

(107)

Since \(\frac{\partial \Delta}{\partial R_Q^*} < 0\) on \(R_Q^* \in (0, L + \frac{2L}{1+\mu} - 4\tau)\), this implies \(\Delta\) is minimized at \(R_Q^* = L + \frac{2L}{1+\mu} - 4\tau\).

For any \(\rho \in (0, 1]\):

\[
\exists \text{ a solution } \frac{R_Q^*}{\rho} = \frac{E[E(R_Q^*|d_t)]}{\rho} \text{ on } R_Q^* \in (0, L + \frac{2L}{1+\mu} - 4\tau]
\]

(108)

\[
\iff \Delta(R_Q^*)|_{L + \frac{2L}{1+\mu} - 4\tau} = \{E[E(R_Q^*|d_t)] - R_Q^*)|_{L + \frac{2L}{1+\mu} - 4\tau} < 0
\]

(109)

\[
\iff \frac{1 + \mu}{12L} \left(3 \left(\frac{2L}{1+\mu}\right)^2 + 2x_1L - x_1^2\right) - \frac{L}{\rho} - \frac{2L/\rho}{1+\mu} + \frac{4\tau}{\rho} < 0.
\]

(110)

We next examine \(\Delta(R_Q^*)|_{L + \frac{2L}{1+\mu} - 4\tau}\) to see if the last inequality holds:

\[
\frac{1 + \mu}{12L} \left(3 \left(\frac{2L}{1+\mu}\right)^2 + 2x_1L - x_1^2\right) - \frac{L}{\rho} - \frac{2L/\rho}{1+\mu} + \frac{4\tau}{\rho}
\]

(111)

\[
= \frac{L}{1+\mu} + \frac{(1 - \mu)x_1}{6} - \frac{(1 + \mu)x_1^2}{12L} - \frac{L}{\rho} - \frac{2L/\rho}{1+\mu} + \frac{4\tau}{\rho}
\]

(112)

\[
= \frac{-L}{\rho} \left(1 + \frac{2 - \rho}{1+\mu}\right) + \frac{(1 + \mu)x_1}{6} \left(1 - \frac{x_1}{2L}\right) + \frac{4\tau}{\rho}
\]

(113)

\[
< \frac{4\tau - L}{\rho} - \frac{L}{1+\mu} + \frac{(1 + \mu)x_1}{6} < 4\tau - L - \frac{L}{1+\mu} + \frac{1 + \mu}{6} L
\]

(114)

\[
< \frac{4\tau - L}{\rho} + L \left(-\frac{1}{2} + \frac{1}{3}\right) < 0.
\]

(115)

Thus there is a unique solution. \(\Box\)

**Lemma 6.** For \(\mu_0 \leq \mu \leq 1\),

\[
\frac{\partial Z_1(\mu)}{\partial \mu} < 0 \quad \text{where} \quad \mu_0 = \frac{2\tau^2}{(L - 2\tau)^2 - 2\tau^2} \quad \text{and}
\]

(116)
\( Z_1(\mu) = \alpha^2(\mu) \left( \frac{\sqrt{2\mu + 2\mu^2}}{\mu(\sqrt{2\mu + 2\mu^2} + 2\mu)} + 1 \right) \).

**Proof.**

\[ \frac{\partial}{\partial \mu} \left[ \alpha^2(\mu) \left[ 1 + \frac{1}{\mu} - \frac{2}{\sqrt{2\mu + 2\mu^2} + 2\mu} \right] \right] = \frac{2\alpha(\mu)4\mu}{(\sqrt{2\mu + 2\mu^2} + 2\mu)^2} \left( \frac{2+4\mu}{\sqrt{2\mu + 2\mu^2} + 2\mu} + 4 \right) \]

\[ + \alpha^2(\mu) \left[ \frac{2+4\mu}{\sqrt{2\mu + 2\mu^2} + 2\mu} + 4 \right] \frac{1}{\mu^2} \]

\[ = \frac{\alpha(\mu)}{\sqrt{2\mu + 2\mu^2} + 2\mu} \left[ \frac{8(1 + \mu)}{(\sqrt{2\mu + 2\mu^2} + 2\mu)^2} - \frac{4}{\mu} \right] \]

\[ - \frac{8\mu - 16\mu^2}{(\sqrt{2\mu + 2\mu^2} + 2\mu)^2(2\mu + 2\mu^2)} + \frac{16\mu}{(\sqrt{2\mu + 2\mu^2} + 2\mu)^2} \]

\[ = \frac{\alpha^2(\mu)}{4\mu} \left[ \frac{8(1 + \mu)\mu(\sqrt{2\mu + 2\mu^2} + 2\mu) - 8\mu^2 + 16\mu^3}{\mu(2\mu + 2\mu^2 + 2\mu)^2\sqrt{2\mu + 2\mu^2}} \right] \]

\[ - \frac{4(2\mu + 2\mu^2 + 2\mu)^2\sqrt{2\mu + 2\mu^2} + 16\mu^2 + 16\mu^2 + \sqrt{2\mu + 2\mu^2}}{\mu(2\mu + 2\mu^2 + 2\mu)^2\sqrt{2\mu + 2\mu^2}} \]

\[ = \frac{\alpha^2(\mu)}{16\mu} \left[ \frac{2\mu\sqrt{2\mu + 2\mu^2} - 4\mu^2 + 2\mu^2\sqrt{2\mu + 2\mu^2} + 8\mu^3}{\mu(2\mu + 2\mu^2 + 2\mu)^2\sqrt{2\mu + 2\mu^2}} \right] \]

\[ - \frac{2\mu\sqrt{2\mu + 2\mu^2} + 2\mu^2\sqrt{2\mu + 2\mu^2} + 4\mu(2\mu + 2\mu^2)}{\mu(2\mu + 2\mu^2 + 2\mu)^2\sqrt{2\mu + 2\mu^2}} \]

\[ = -12\mu^2 < 0 \forall \mu > 0. \]

**Proof of Proposition 10.**

\[ \frac{E[E(R_Q^\circ|d_i)]}{\rho} = \frac{R_Q^\circ}{\rho} \]

\[ \iff \frac{1 + \mu}{12L} \left[ 3(L - R_Q^\circ - 4\tau)^2 + L^2 - \alpha^2(\mu)L^2 \right] = \frac{R_Q^\circ}{\rho}. \]
Differentiating both sides wrt $\mu$ yields

\[
\frac{1}{12L} \left[ 3(L - R_Q^o - 4\tau)^2 + L^2 - \alpha^2(\mu) L^2 \right]
\]

\[
\frac{1 + \mu}{12L} \left[ 6(L - R_Q^o - 4\tau) \left( -\frac{\partial R_Q^o}{\partial \mu} \right) - 2\alpha(\mu) \frac{\partial \alpha(\mu)}{\partial \mu} L^2 \right] = \frac{\partial R_Q^o}{\rho \partial \mu}
\]

\[
\iff \quad \left( \frac{1}{\rho} + \frac{1 + \mu}{2L} (L - R_Q^o - 4\tau) \right) \frac{\partial R_Q^o}{\partial \mu}
\]

\[
L \left[ 3 \left( 1 - \frac{R_Q^o + 4\tau}{L} \right)^2 + 1 - \alpha^2(\mu) - 2(1 + \mu) \alpha(\mu) \frac{\partial \alpha(\mu)}{\partial \mu} \right]
\]

\[
= \frac{12}{12}
\]

\[
L \left[ 3 \left( 1 - \frac{R_Q^o + 4\tau}{L} \right)^2 + 1 - \alpha^2(\mu) \left( \frac{\sqrt{2\mu + 2\mu^2}}{\mu(2\mu + 2\mu^2 + 2\mu)} + 1 \right) \right]
\]

\[
= \frac{12}{12}
\]

The final summand is $Z_1(\mu)$ from Lemma 6. Since $\frac{\partial Z_1(\mu)}{\partial \mu} < 0$ for $\mu_0 \leq \mu \leq 1$, we have that $Z_1(\mu_0) \geq Z_1(\mu) \geq Z_1(1) = \frac{3}{2}$ where $Z_1(\mu_0) = \frac{4\tau}{L}$. Then

\[
Z_1(\mu_0) = \alpha^2(\mu_0) \left( \frac{1}{\mu_0(1 + \frac{2}{\sqrt{2/\mu_0} + 2})} + 1 \right)
\]

\[
= \frac{16\tau^2}{L^2} \left( \frac{(L - 2\tau)^2 - 2\tau^2}{2\tau^2 \left( 1 + \frac{2\tau}{L - 2\tau} \right)} + 1 \right)
\]

\[
= 8 \left( 1 - 6\frac{\tau}{L} + 12 \left( \frac{\tau}{L} \right)^2 - 4 \left( \frac{\tau}{L} \right)^3 \right)
\]

makes it easy to see that $\frac{3}{2} \leq Z_1(\mu_0) \leq 8$.

Note that $3 \left( 1 - \frac{R_Q^o + 4\tau}{L} \right)^2 + 1 \in (1, 4)$. Also note that $R_Q^o$ depends on both $\mu$ and $\rho$. In particular, when $\rho$ is small enough, $R_Q^o$ is very close to zero. Therefore, the sign of $\frac{\partial R_Q^o}{\partial \mu}$ is not the same for all $\mu$ (i.e. there is no uniform conclusion).

\[\square\]

**Proof of Corollary 3.** The proof is simple: Note that

\[
Z(\mu) < 3 \left( \frac{1}{\sqrt{6}} \right)^2 + 1 - Z_1(1) = \frac{3}{2} - \frac{3}{2} = 0.
\]

\[\square\]
Proof of Corollary 4.

\begin{equation}
R_Q^\circ + 4\tau < \left(1 - \frac{1}{\sqrt{6}}L\right) \Rightarrow 3 \left(1 - \frac{R_Q^\circ + 4\tau}{L}\right) + 1 > \frac{3}{2}
\end{equation}

Since \(\min_\mu Z_1(\mu) = Z_1(1) = \frac{3}{2}\), when \(\mu\) is close to 1:

\begin{equation}
Z_1(\mu) < 3 \left(1 - \frac{R_Q^\circ + 4\tau}{L}\right) + 1.
\end{equation}

\[\Box\]

Appendix D. Solution of Normal with Market Makers Case

**Lemma 7.** The function \(f(x) = (x + c)(1 - \Phi(ax + b)) + d\) for \(a > 0\) has a unique global maximum on \(\mathbb{R}\).

**Proof.**

\begin{equation}
\frac{\partial f}{\partial x} = -a(x + c)\phi(ax + b) + 1 - \Phi(ax + b)
\end{equation}

\begin{equation}
\frac{\partial^2 f}{\partial x^2} = \phi(ax + b)(-a + a^2(x + c)(ax + b) - a)
\end{equation}

\begin{equation}
= a\phi(ax + b)(-2 + a^2x^2 + a(b + ac)x + abc).
\end{equation}

\(\frac{\partial^2 f}{\partial x^2}\) has roots at

\begin{equation}
x_0 = \frac{-ab - a^2c \pm \sqrt{a^2(b + ac)^2 - 4a^2(abc - 2)}}{2a^2}
\end{equation}

\begin{equation}
= \frac{-b}{2a} - \frac{c}{2} \pm \frac{\sqrt{(b - ac)^2 + 8}}{2a}.
\end{equation}

The two roots are real (one positive, one negative) since the interior of the square root is positive. For \(x < c\), \(f\) is negative; for \(x > c\), \(f\) is positive. However, for \(x\) greater than the largest root \(x > \frac{-b}{2a} - \frac{c}{2} + \frac{\sqrt{(b - ac)^2 + 8}}{2a}\), \(\frac{\partial f}{\partial x} < 0\) and \(\frac{\partial^2 f}{\partial x^2} > 0\). Thus \(f\) descends to an asymptote.

Since \(f\) is continuous and \(\frac{\partial f}{\partial x^2} < 0\) for \(x\) between the two roots, \(f\) attains a global maximum on \((-\frac{b - ac}{2a}, \frac{-b - ac}{2a} + \frac{\sqrt{(b - ac)^2 + 8}}{2a^2})\). \[\Box\]
**Proof of Proposition 11.** Since $R_Q|d_t$ contains two indicator functions, we consider three regions: where neither is active, where the first indicator is active, and where the second indicator is active. By Proposition 1, we know the indicator functions are never both active.

By Lemma 7, we can find a global maximum on an unconstrained interval. We find maxima for each constrained interval and thus the max of $R_Q|d_t$ over the three regions. We then compute the unconditional expected value of $R_Q^*|d_t$ over the $d_t$ distribution (common knowledge), yielding $E[E(R_Q^*|d_t)] = R_Q^o$. Since the maximum is unique, we have a unique Markov Perfect equilibrium. □

**Proof of Proposition 12.** If the distribution of reservation values $F$ is Gaussian, then the bid-ask spread is given by

$$
\delta + \beta = \frac{\Phi((-R_Q^o - \delta - 2\tau)/L)}{\phi((-R_Q^o - \delta - 2\tau)/L)} + \frac{\Phi((-R_Q^o - \beta - 2\tau)/L)}{\phi((-R_Q^o - \beta - 2\tau)/L)} + 4\tau.
$$

From Lemma 2 in Feller I (p. 175), we get that:

$$
\delta + \beta \leq \frac{L}{|R_Q^o + \delta + 2\tau|} + \frac{L}{|R_Q^o + \beta + 2\tau|} + 4\tau.
$$

Using Jensen’s inequality, we get

$$
\delta + \beta \leq \frac{2L}{2R_Q^o + \delta + \beta + 4\tau} + 4\tau \leq \frac{L}{R_Q^o + 4\tau} + 4\tau.
$$

□

**References**


