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Hedging American contingent claims with constrained portfolios

Ioannis Karatzas[1,*], **S. G. Kou**[2]

[1]Departments of Mathematics and Statistics, Columbia University, New York, NY 10027, USA
(e-mail: ik@math.columbia.edu)
[2]Department of Statistics, University of Michigan, Mason Hall, Ann Arbor, MI 48109-1027, USA
(e-mail: kou@stat.umich.lsa.umich.edu)

Abstract. The valuation theory for American Contingent Claims, due to Bensoussan (1984) and Karatzas (1988), is extended to deal with *constraints on portfolio choice*, including incomplete markets and borrowing/short-selling constraints, or with *different interest rates* for borrowing and lending. In the unconstrained case, the classical theory provides a single arbitrage-free price u_0; this is expressed as the supremum, over all stopping times, of the claim's expected discounted value under the equivalent martingale measure. In the presence of constraints, $\{u_0\}$ is replaced by an entire interval $[h_{\text{low}}, h_{\text{up}}]$ of arbitrage-free prices, with endpoints characterized as $h_{\text{low}} = \inf_{\nu \in \mathscr{D}} u_\nu,\ h_{\text{up}} = \sup_{\nu \in \mathscr{D}} u_\nu$. Here u_ν is the analogue of u_0, the arbitrage-free price with unconstrained portfolios, in an auxiliary market model $\mathscr{M}_\nu$; and the family $\{\mathscr{M}_\nu\}_{\nu \in \mathscr{D}}$ is suitably chosen, to contain the original model and to reflect the constraints on portfolios. For several such constraints, explicit computations of the endpoints are carried out in the case of the American call-option. The analysis involves novel results in martingale theory (including simultaneous Doob-Meyer decompositions), optimal stopping and stochastic control problems, stochastic games, and uses tools from convex analysis.

Key words: Contingent claims, hedging, pricing, arbitrage, constrained markets, incomplete markets, different interest rates, Black-Scholes formula, optimal stopping, free boundary, stochastic control, stochastic games, equivalent martingale measures, simultaneous Doob-Meyer decompositions.

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1. Introduction

The purpose of this paper is to present a theory for the hedging of American Contingent Claims (ACCs), under constraints on portfolio-choice which include

(i) prohibition of (or constraints on) borrowing,
(ii) prohibition of (or constraints on) short-selling of stocks,
(iii) prohibition of investment in some particular stocks (i.e., incomplete markets),

as well as in the presence of a higher interest rate for borrowing than for saving. "American" Contingent Claims, such as American call- or put-options, differ from their "European" counterparts in that they can be exercised by their holder at any time $0 \le t \le T$ during a given horizon $[0, T]$, where T is the so-called "maturity" of the claim; in contrast, "European" contingent claims can be exercised only at maturity ($t = T$). It is this extra feature that makes the valuation problem for the ACCs more interesting and, of course, more challenging.

The hedging problem for European Contingent Claims (ECCs), in a complete market and *without constraints* on portfolio choice, is by now well-understood; its theory begins with the seminal papers of Black and Scholes (1973) and Merton (1973), and "matures" with the work of Ross (1976), Harrison and Kreps (1979), Harrison and Pliska (1981, 1983) through which the connections with arbitrage and with the equivalent martingale measure are made explicit. The pricing of ECCs *under constraints* on portfolio choice (which include incomplete markets as a special case) was developed by Cvitanić and Karatzas (1993) and by Karatzas and Kou (1996) through a mixture of probabilistic and analytical techniques. Related work by Ansel and Stricker (1992), Bergman (1995), El Karoui and Quenez (1991, 1995), Jouini and Kallal (1993), Korn (1992), Naik and Uppal (1994), Föllmer and Kramkov (1995), Kramkov (1996) treated particular aspects of similar problems, in various degrees of generality.

On the other hand, the valuation theory for American Contingent Claims with *unconstrained portfolios* goes back to Samuelson (1965), to McKean (1965) who treated formally the valuation problem for the American call-option on a dividend-paying stock as a question in optimal stopping and solved the associated free-boundary problem, and to Merton (1973); see our Theorem 7.2, and Smith (1976), for a survey of this early work. This formal theory was given "financial" justification by Bensoussan (1984) and Karatzas (1988), based on hedging arguments and using explicitly the equivalent martingale measure methodologies of Harrison and Pliska (1981); see the survey paper by Myneni (1992), as well as Karatzas (1997, Sect. 1.4).

We review this theory in Sects. 2 and 3; even within the "classical" setup we present a novel approach because we distinguish clearly the roles of the seller and the buyer, which are quite asymmetric in the context of ACCs. This asymmetry reflects itself in the definitions of the upper- and lower-hedging prices in (3.3) and (3.5), respectively. The main result is that, in a complete market and without constraints on portfolio choice, the upper- and lower-hedging prices are equal, and are given by the maximal expected reward $u(0)$, under the equivalent martingale measure, in an optimal stopping problem involving the claim's discounted value (Theorem 3.3); and this common value also gives the unique arbitrage-free price for the ACC. We present in Sect. 3 some of the standard examples on the American call- and put-options that can be solved explicitly, both for completeness of exposition and for later usage.

In Sect. 4 we formulate the hedging problem for the ACC under general *portfolio constraints*. The upper- and lower-hedging prices, $h_{\text{up}}(K)$ and $h_{\text{low}}(K)$, respectively, are extended to this new context (Definitions (4.4) and (4.5)), the notion of "arbitrage opportunity" is introduced (Definition 4.2), and it is shown that the interval

$$\left[h_{\text{low}}(K), h_{\text{up}}(K)\right] \tag{1.1}$$

is the largest one can obtain based on arbitrage considerations alone: no price inside this interval leads to an arbitrage opportunity, while every price outside the interval does (Theorem 4.3). Equivalently, the effect of constraints is to "enlarge" the set of arbitrage-free prices, from the singleton $\{u(0)\}$ of the unconstrained case, to the interval (1.1) which contains $u(0)$ (Lemma 3.1 and (4.6)). Similar results for European Contingent Claims were originated by Karatzas and Kou (1996).

Section 5 contains the main results of the paper. We specialize there to convex constraints on portfolio and show, in that context, how to compute the upper- and lower-hedging prices of (1.1). As in Cvitanić and Karatzas (1992, 1993) and Karatzas and Kou (1996), we introduce an auxiliary family $\mathscr{M}_\nu,\ \nu \in \mathscr{D}$ of unconstrained markets (random environments) – each with its own equivalent martingale measure $\mathbb{P}_\nu$, discount factor $\gamma_\nu(\cdot)$, and hedging price

$$u_\nu(0) = \sup_{\tau \in \mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau)B(\tau)], \quad \nu \in \mathscr{D} \tag{1.2}$$

for the ACC $B(\cdot)$ with unconstrained portfolios, where $\mathscr{S}$ is the class of stopping times τ with values in the interval $[0, T]$. Our original market-model is a member of this family, and this latter is designed so as to reflect the convex portfolio constraints. It turns out that the upper- and lower-hedging prices are given by the supremum and the infimum, respectively, of the quanities in (1.2), over the family of all these auxiliary random environments:

$$h_{\text{up}}(K) = \sup_{\nu \in \mathscr{D}} u_\nu(0) = \sup_{\nu \in \mathscr{D}} \sup_{\tau \in \mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau)B(\tau)], \tag{1.3}$$

$$h_{\text{low}}(K) = \inf_{\nu \in \mathscr{D}} u_\nu(0) = \inf_{\nu \in \mathscr{D}} \sup_{\tau \in \mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau)B(\tau)]. \tag{1.4}$$

The justification of the representations (1.3), (1.4) is carried out in the Appendices A and B, respectively; it turns out to be quite demanding, as it involves not only optimal stopping, but also novel problems in stochastic control and stochastic games. "Simultaneous Doob-Meyer decompositions," valid under a whole family of probability measures, also play an important role in the analysis, as they did in

El Karoui and Quenez (1991, 1995), Cvitanić and Karatzas (1993), and Karatzas and Kou (1996); see also Föllmer and Kramkov (1995), Kramkov (1996).

Sections 6 and 7 use the representations (1.3), (1.4) to *compute* the hedging prices in a variety of examples, involving the American call-option. Section 8 treats briefly the case of different interest rates for borrowing and saving.

2. The model

We shall deal in this paper with the following standard model for a financial market $\mathscr{M}$ with $d+1$ assets, which can be traded continuously. One of these assets, called the *bond* (or "bank account"), has price $S_0(\cdot)$ governed by

$$dS_0(t) = S_0(t)r(t)dt, \quad S_0(0) = 1. \tag{2.1}$$

The remaining d assets are subject to systematic risk; we shall refer to them as *stocks*, and assume that the price-per-share $S_i(\cdot)$ of the ith stock is modelled by the equation

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right], \quad S_i(0) = s_i \in (0,\infty) \tag{2.2}$$

for every $i = 1, \ldots, d$.

In this model $\mathscr{M}$, the components of the d-dimensional *Brownian motion* $W(t) = (W_1(t), \ldots, W_d(t))^*$, $0 \le t \le T$, model the independent sources of systematic risk, and $\sigma_{ij}(t)$ is the intensity with which the jth source of uncertainty influences the price of the ith stock at time $t \in [0,T]$. Here $T > 0$ is the *time-horizon* of the model; unless explicitly stated otherwise, it will be assumed finite. The Brownian motion $W(\cdot)$ is defined on the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$; the augmentation of its natural filtration $\mathscr{F}^W(t) = \sigma(W(s),\ 0 \le s \le t)$, $\quad 0 \le t \le T$ will be denoted throughout by $\mathbb{F} = \{\mathscr{F}(t)\}_{0\le t \le T}$. The processes $r(t)$, $0 \le t \le T$ (the *interest rate*), $b(t) = (b_1(t), \ldots, b_d(t))^*$, $0 \le t \le T$ (the *vector of stock appreciation rates*) and $\sigma(t) = (\sigma_{ij}(t))_{1\le i,j\le d}$, $0 \le t \le T$ (the *volatility matrix*) are the "coefficients" of this model. They will be assumed throughout to be $\mathbb{F}$-progressively measurable, and bounded uniformly in $(t,w) \in [0,T] \times \Omega$; in addition, $\sigma(t,w)$ will be assumed to be invertible, with $\sigma^{-1}(t,w)$ bounded uniformly in $(t,w) \in [0,T] \times \Omega$.

Under these assumptions, the *relative risk* process of $\mathscr{M}$, namely

$$\theta(t) \stackrel{\Delta}{=} \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}], \quad 0 \le t \le T, \tag{2.3}$$

is bounded and $\mathbb{F}$-progressively measurable; thus

$$Z_0(t) \stackrel{\Delta}{=} \exp\left[-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right], \quad 0 \le t \le T \tag{2.4}$$

is a martingale, and

(2.5)
$$W^{(0)}(t) \overset{\Delta}{=} W(t) + \int_0^t \theta(s)ds, \quad 0 \le t \le T$$

is a Brownian motion under the probability measure

(2.6)
$$\mathbb{P}^0(A) \overset{\Delta}{=} \mathbb{E}[Z_0(T)1_A], \quad A \in \mathscr{F}(T),$$

by the Girsanov theorem (e.g. Karatzas and Shreve (1991), Sect. 3.5).

2.1 Remark: The probability measure $\mathbb{P}^0$ of (2.6) is called *risk-neutral equivalent martingale measure*; it is equivalent to $\mathbb{P}$, and it is clear from (2.3), (2.5) that we may rewrite (2.2) in the form

(2.7)
$$dS_i(t) = S_i(t)\left[r(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j^{(0)}(t)\right], \quad S_i(0) = s_i \in (0,\infty)$$

or equivalently $d(\gamma_0(t)S_i(t)) = (\gamma_0(t)S_i(t)) \cdot \sum_{j=1}^d \sigma_{ij}(t)dW_j^{(0)}$, where we have set

(2.8)
$$\gamma_0(t) \overset{\Delta}{=} \frac{1}{S_0(t)} = \exp\left(-\int_0^t r(s)ds\right), \quad 0 \le t \le T.$$

In other words, under $\mathbb{P}^0$ the discounted stock prices $\gamma_0(\cdot)S_j(\cdot)$, $i = 1, \dots, d$ are martingales.

2.2 Definition: *(i) An $\mathbb{F}$-progressively measurable process $\pi : [0,T] \times \Omega \to \mathscr{R}^d$ with $\int_0^T \|\pi(t)\|^2 dt < \infty$ a.s., is called* **portfolio process***.*

(ii) An $\mathbb{F}$-adapted process $C : [0,T] \times \Omega \to [0,\infty)$ with increasing, right-continuous paths and $C(0) = 0$, $C(T) < \infty$ a.s., is called **cumulative consumption** *process.* ▫

2.3 Definition: *For any given portfolio/cumulative consumption process pair (π, C), and $x \in \mathscr{R}$, the solution $X(\cdot) \equiv X^{x,\pi,C}(\cdot)$ of the linear stochastic equation*

$$\begin{aligned}
dX(t) &= \sum_{i=1}^d \pi_i(t) \cdot \frac{dS_i(t)}{S_i(t)} + \left(X(t) - \sum_{i=1}^d \pi_i(t)\right) \cdot \frac{dS_0(t)}{S_0(t)} - dC(t) \\
&= \sum_{i=1}^d \pi_i(t)\left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t)\right] \\
&\quad + \left(X(t) - \sum_{i=1}^d \pi_i(t)\right) r(t)dt - dC(t)
\end{aligned}$$

(2.9)
$$= r(t)X(t)dt + \pi^*(t)\sigma(t)dW^{(0)}(t) - dC(t), \quad X(0) = x,$$

is called the **wealth process** *corresponding to initial capital x, portfolio rule $\pi(\cdot)$, and cumulative consumption rule $C(\cdot)$.* ▫

The interpretation of these quantities should be clear: $\pi_i(t)$ represents the amount of the agent's wealth that is invested in the ith stock at time t, and this amount may be positive or negative, which means that short-selling of stocks is permitted. The amount $X(t) - \sum_{i=1}^{d} \pi_i(t)$ not invested in stocks is put into the bank-account, and it too is allowed to take negative values (corresponding to borrowing rather than saving, at the interest rate $r(t)$). In particular, the vector processes $p(t) = (p_1(t), \ldots, p_d(t))^*$, and $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))^*$, $0 \leq t \leq T$, with entries

$$p_i(t) \triangleq \left\{ \begin{array}{ll} \pi_i(t)/X(t) & ; X(t) \neq 0 \\ 0 & ; X(t) = 0 \end{array} \right\}, \quad i = 1, \ldots, d,$$

$$\varphi_i(t) \triangleq \left\{ \begin{array}{ll} \pi_i(t)/S_i(t) = (X(t)p_i(t))/S_i(t) & ; i = 1, \ldots, d \\ \frac{X(t) - \sum_{j=1}^{d} \pi_j(t)}{S_0(t)} = \frac{X(t)[1 - \sum_{j=1}^{d} p_j(t)]}{S_0(t)} & ; i = 0 \end{array} \right\} \tag{2.10}$$

provide, respectively, the *proportions of wealth* and the *number-of-shares* held in each of the assets at time t, and we obtain

$$X(t) = \sum_{i=0}^{d} \varphi_i(t) S_i(t), \quad \forall\ 0 \leq t \leq T. \tag{2.11}$$

On the other hand, $C(t+h) - C(t)$ represents the amount withdrawn (for "consumption") during the interval $(t, t+h)$, $h > 0$. Finally, let us notice that the solution of (2.9) is given by

$$\begin{aligned} \gamma_0(t)X(t) + \int_{(0,t]} \gamma_0(s)dC(s) & \\ = x + \int_0^t \gamma_0(s)\pi^*(s)\sigma(s)dW^{(0)}(s), & \\ = x + \int_0^t \gamma_0(s)X(s)p^*(s)\sigma(s)dW^{(0)}(s), & \quad 0 \leq t \leq T. \end{aligned} \tag{2.12}$$

2.4 Definition: *We say that a portfolio/consumption process pair* (π, C) *as in Definitions 2.2 and 2.3, is* **admissible in $\mathscr{M}$ for the initial wealth** x, *if there exists a nonnegative random variable* Λ *with* $\mathbb{E}^0(\Lambda^p) < \infty$ *for some* $p > 1$, *such that the wealth process* $X(\cdot) \equiv X^{x,\pi,C}(\cdot)$ *of (2.9), (2.12) satisfies almost surely:*

$$X^{x,\pi,C}(t) \geq -\Lambda, \quad \forall\ 0 \leq t \leq T. \tag{2.13}$$

We shall denote by $\mathscr{A}_0(x)$ *the class of all such pairs.* □

The requirement of admissibility is imposed, in order to rule out "doubling strategies" (cf. Harrison and Pliska (1981), Karatzas and Shreve (1998)); these achieve arbitrarily large levels of wealth, but violate the condition of Definition 2.4. In particular, if $X(\cdot)$ is a.s. bounded from below on $[0, T]$ as in (2.13), then the process of (2.12) is a $\mathbb{P}^0$-local martingale and bounded from below, thus a supermartingale under $\mathbb{P}^0$, and the optional sampling theorem gives

$$\mathbb{E}^0\left[\gamma_0(\tau)X(\tau) + \int_{(0,\tau]} \gamma_0(t)dC(t)\right] \leq x; \quad \forall\ \tau \in \mathscr{S}, \quad \forall\ (\pi, C) \in \mathscr{A}_0(x, \tau). \tag{2.14}$$

Here and in the sequel, we are denoting by $\mathscr{S}_{s,t}$ the class of $\mathbb{F}$-stopping times $\tau : \Omega \to [s,t]$, for $0 \le s \le t \le T$, and let $\mathscr{S} \equiv \mathscr{S}_{0,T}$.

2.5 Remark on notation: For any given $\tau \in \mathscr{S}$ we denote, in (2.14) and in the sequel, by $\mathscr{A}_0(x,\tau)$ the class of portfolio/consumption process pairs (π, C) for which the *stopped process* $X^{x,\pi,C}(\cdot \wedge \tau)$ satisfies the requirement (2.13). Clearly, $\mathscr{A}_0(x) = \mathscr{A}_0(x,T) \subseteq \mathscr{A}_0(x,\tau),\ \forall \tau \in \mathscr{S}$.

3. American contingent claims in an unconstrained market

Let us consider now the following situation: two agents enter at time $t = 0$ into an agreement. One of them (the "seller") agrees to provide to the second agent (the "buyer") a random amount $B(\tau(\omega),\omega) \ge 0$ at time $t = \tau(\omega)$, where $\tau : \Omega \to [0,T]$ is a stopping time of $\mathbb{F}$ and *at the disposal of the buyer*. We shall assume throughout that $B : [0,T] \times \Omega \to [0,\infty)$ is an $\mathbb{F}$-adapted process with continuous paths and

$$\mathbb{E}\left[\sup_{0 \le t \le T} (\gamma_0(t) B(t))^{1+\epsilon}\right] < \infty, \quad \text{for some } \epsilon > 0. \tag{3.1}$$

In return for this commitment, the buyer agrees to pay an amount $x \ge 0$ to the seller at time $t = 0$. *What should this amount be?* In other words, what is the "fair price" to pay at $t = 0$ to the seller, for his obligation — to deliver the amount $B(\tau) \ge 0$ to the buyer at a stopping time $\tau \in \mathscr{S}$ of the buyer's choice?

A process $B(\cdot)$ with the properties stated above is called an *American Contingent Claim* (ACC); and the question we just posed, is the *pricing problem* for this American Contingent Claim. The "classical" examples are the American *call-option* $B(t) = (S_i(t) - q)^+$ and the American *put-option* $B(t) = (q - S_i(t))^+$, $0 \le t \le T$ on the i^{th} stock, with exercise price $0 \le q < \infty$. As we shall see in this section, the pricing problem admits a complete solution in the framework of the model $\mathscr{M}$ of (2.1), (2.2).

To tackle the pricing problem, one has to look at the situation of each agent separately. The *seller's objective* is, starting with the amount $x \ge 0$ that he receives from the buyer at $t = 0$, to find a portfolio/consumption process pair $(\hat{\pi}, \hat{C})$ that makes it possible for him to fulfil his obligation *without risk* (i.e., with probability one) and *whenever the buyer should choose to ask for the payment:*

$$X^{x,\hat{\pi},\hat{C}}(\tau) \ge B(\tau) \text{ a.s.}, \quad \forall\, \tau \in \mathscr{S}. \tag{3.2}$$

The smallest value of initial capital $x \ge 0$ that allows the seller to do this, is called *upper hedging price* for the ACC:

$$h_{\text{up}} \stackrel{\Delta}{=} \inf\{x \ge 0 / \exists (\hat{\pi}, \hat{C}) \in \mathscr{A}_0(x) \text{ s.t. (3.2) holds}\}. \tag{3.3}$$

Consider now the *buyer's objective:* he starts out with the amount $-x$ (as he pays $x \ge 0$ to the seller) at time $t = 0$, and looks for a stopping time $\check{\tau} \in \mathscr{S}$, and a portfolio/consumption strategy $(\check{\pi}, \check{C}) \in \mathscr{A}_0(-x, \check{\tau})$, such that, by exercising his option at time $t = \check{\tau}(w)$, the payment that he receives allows him to recover the debt he incurred at $t = 0$ by purchasing the ACC:

$$X^{-x,\check{\pi},\check{C}}(\check{\tau}) + B(\check{\tau}) \geq 0, \text{ a.s.} \tag{3.4}$$

The largest amount $x \geq 0$ that enables the buyer to do this, is called *lower hedging price* for the ACC:

$$h_{\text{low}} \stackrel{\Delta}{=} \sup\{x \geq 0 / \exists \check{\tau} \in \mathscr{S}, (\check{\pi}, \check{C}) \in \mathscr{A}_0(-x, \check{\tau}) \text{ s.t. (3.4) holds}\}. \tag{3.5}$$

The reader should not fail to notice the *asymmetry* in the definitions of the upper and lower hedging prices in (3.3), (3.5), respectively. This asymmetry reflects the fundamental asymmetry in the situations of the seller and the buyer: the former needs to hedge against *any* stopping time $\tau \in \mathscr{S}$ in (3.2), whereas the latter need only hedge as in (3.4) for *some* stopping time $\check{\tau} \in \mathscr{S}$.

The following inequality (3.7) justifies the terminology "upper" and "lower" hedging price.

3.1 Lemma: *Consider the decreasing function*

$$u(t) \stackrel{\Delta}{=} \sup_{\tau \in \mathscr{S}_{t,T}} \mathbb{E}^0[\gamma_0(\tau)B(\tau)], \quad 0 \leq t \leq T. \tag{3.6}$$

We have

$$0 \leq B(0) \leq h_{\text{low}} \leq u(0) \leq h_{\text{up}} \leq \infty. \tag{3.7}$$

Proof: If the set of (3.3) is empty, then $h_{\text{up}} = \infty$ and $h_{\text{up}} \geq u(0)$ holds trivially; if not, let x be an arbitrary element of this set, and observe from (2.14), (3.2) that we have:

$$x \geq \mathbb{E}^0\left[\gamma_0(\tau)X^{x,\hat{\pi},\hat{C}}(\tau) + \int_{(0,\tau]} \gamma_0(t)d\check{C}(t)\right] \geq \mathbb{E}^0[\gamma_0(\tau)B(\tau)]; \quad \forall\, \tau \in \mathscr{S}.$$

Thus $x \geq u(0)$, and $h_{\text{up}} \geq u(0)$ follows from the arbitrariness of x. On the other hand, the number $B(0)$ clearly belongs to the set of (3.5) (just take $x = B(0) \geq 0$, $\check{\tau} = 0$, $\check{\pi}(\cdot) \equiv 0$, $\check{C}(\cdot) \equiv 0$ in (3.4)); for an arbitrary element $x \geq 0$ of this set, (2.14) and (3.4) give

$$-x \geq \mathbb{E}^0\left[\gamma_0(\check{\tau})X^{-x,\check{\pi},\check{C}}(\check{\tau}) + \int_{(0,\check{\tau}]} \gamma_0(t)d\check{C}(t)\right] \geq -\mathbb{E}^0[\gamma_0(\check{\tau})B(\check{\tau})] \geq -u(0),$$

whence $h_{\text{low}} \leq u(0)$ from the arbitrariness of x.

3.2 Remark: From condition (3.1), and the boundedness of the process $\theta(\cdot)$ in (2.3), we obtain

$$\mathbb{E}^0\left[\sup_{0\le t\le T}(\gamma_0(t)B(t))\right] = \mathbb{E}\left[Z_0(T)\cdot\sup_{0\le t\le T}(\gamma_0(t)B(t))\right]$$
$$\le \left(\mathbb{E}(Z_0(T))^q\right)^{1/q}\cdot\left(\mathbb{E}\sup_{0\le t\le T}(\gamma_0(t)B(t))^p\right)^{1/p} < \infty$$

with $p = 1+\epsilon > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. In particular, $u(0) < \infty$ in (3.6).

Here is the main theoretical result of this section.

3.3 Theorem. *The infimum of (3.3) and the supremum of (3.5) are both attained, and are equal:*

(3.8) $$h_{\text{up}} = h_{\text{low}} = u(0) \stackrel{\Delta}{=} \sup_{\tau\in\mathscr{S}}\mathbb{E}^0[\gamma_0(\tau)B(\tau)] < \infty.$$

Furthermore, there exists a pair $(\hat{\pi},\hat{C}) \in \mathscr{A}_0(u(0))$ *such that, with*

(3.9) $$\hat{X}_0(t) \stackrel{\Delta}{=} \frac{1}{\gamma_0(t)}\operatorname{ess\,sup}_{\tau\in\mathscr{S}_{t,T}}\mathbb{E}^0[\gamma_0(\tau)B(\tau)|\mathscr{F}(t)], \quad 0\le t\le T,$$

(3.10) $$\check{\tau} \stackrel{\Delta}{=} \inf\{t\in[0,T)/\hat{X}_0(t) = B(t)\}\wedge T,$$

and $\check{\pi}(\cdot) \equiv -\hat{\pi}(\cdot)$, *we have almost surely:*

(3.11) $$X^{u(0),\hat{\pi},\hat{C}}(t) = \hat{X}_0(t) \ge B(t), \quad \forall\ 0\le t\le T,$$

(3.12) $$X^{u(0),\hat{\pi},\hat{C}}(t) = -X^{-u(0),\check{\pi},0}(t) > B(t), \quad \forall\ 0\le t<\check{\tau},$$

(3.13) $$\hat{C}(\check{\tau}) = 0, \quad X^{u(0),\hat{\pi},\hat{C}}(\check{\tau}) = -X^{-u(0),\check{\pi},0}(\check{\tau}) = B(\check{\tau}). \quad \square$$

The portfolio $\hat{\pi}(\cdot)$ (respectively, $\check{\pi}(\cdot)$) is the *optimal hedging portfolio* for the seller (respectively, the buyer). The stopping time of $\check{\tau}$ of (3.10) is the *optimal exercise time* for the buyer; and the process $\hat{X}_0(\cdot)$ of (3.9) is called the *price-process of the ACC* in $[0,T]$.

We shall refer the reader to Karatzas and Shreve (1998, Sect. 2.7) for the proof of Theorem 3.3; see also the survey by Myneni (1992), and Jacka (1991).

Although closed form solutions are typically not available for pricing American options on finite-horizons, an extensive literature exists on their numerical computation. We shall make here no attempt to survey all the existing literature; rather, interested readers are referred to several survey papers and books such as Broadie and Detemple (1994), Boyle et al. (1996), Carverhill and Webber (1990), Hull (1993), Wilmott et al. (1993) for a partial list of fairly recent numerical work on American options and comparisons of efficiency.

4. Constraints on portfolio choice

Let us introduce now *constraints* on the portfolios available to agents. Suppose that two Borel subsets K_+, K_- of $\mathscr{R}^d$ are given, *each of which contains the origin*, and we restrict attention to portfolio/consumption rules (π, C) that satisfy

$$\begin{aligned} p(t) &\in K_+, \quad \text{as long as } X^{x,\pi,C}(t) > 0 \\ p(t) &\in K_-, \quad \text{as long as } X^{x,\pi,C}(t) < 0, \end{aligned}$$

where $p(\cdot)$ is the portfolio-proportion process of (2.10). In other words, our class of admissible portfolio/consumption process pairs becomes now

$$\begin{aligned} \mathscr{A}(x) \stackrel{\Delta}{=} \{(\pi, C) \in \mathscr{A}_0(x)/ \quad & p(t) \in K_+ \text{ on } \{X^{x,\pi,C}(t) \geq 0\}, \text{ and} \\ & p(t) \in K_- \text{ on } \{X^{x,\pi,C}(t) \leq 0\}, \ \forall 0 \leq t \leq T\}. \end{aligned} \tag{4.1}$$

We shall consider also the subclasses

$$\begin{aligned} \mathscr{A}_+(x) \stackrel{\Delta}{=} \{(\pi, C) \in \mathscr{A}(x)/p(t) \in K_+ \text{ and } X^{x,\pi,C}(t) \geq 0, \ \forall 0 \leq t \leq T, \text{ a.s.}\} \\ \text{for } x \geq 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \mathscr{A}_-(x) \stackrel{\Delta}{=} \{(\pi, C) \in \mathscr{A}(x)/p(t) \in K_- \text{ and } X^{x,\pi,C}(t) \leq 0, \ \forall 0 \leq t \leq T, \text{ a.s.}\} \\ \text{for } x \leq 0, \end{aligned} \tag{4.3}$$

and define $\mathscr{A}(x, \tau)$, $\mathscr{A}_\pm(x, \tau)$ for any given $\tau \in \mathscr{S}$, just as in Remark 2.5.

4.1 Remark on notation: We shall denote by $\mathscr{M}(K)$ the market $\mathscr{M}$ of (2.1), (2.2), (2.9), *constrained* by the requirement that portfolio/consumption rules (π, C) should belong to the class $\mathscr{A}(x)$ of (4.1).

Consider now, in this constrained market $\mathscr{M}(K)$, an American Contingent Claim (ACC) $B(\cdot) = \{B(t), \ 0 \leq t \leq T\}$ as in the beginning of Sect. 3. By analogy with (3.3), (3.5) and the discussion preceding them, we can introduce the *upper-hedging price*

$$h_{\text{up}}(K) \stackrel{\Delta}{=} \inf\{x \geq 0/\exists(\hat{\pi}, \hat{C}) \in \mathscr{A}_+(x) \text{ s.t. (3.2) holds}\} \tag{4.4}$$

and the *lower-hedging price*

$$h_{\text{low}}(K) \stackrel{\Delta}{=} \sup\{x \geq 0/\exists \check{\tau} \in \mathscr{S}, \ \exists(\check{\pi}, \check{C}) \in \mathscr{A}_-(-x, \check{\tau}) \text{ s.t. (3.4) holds}\} \tag{4.5}$$

of $B(\cdot)$ *with constrained portfolios*. And just as in Lemma 3.1, we have here as well

$$0 \leq B(0) \leq h_{\text{low}}(K) \leq u(0) \leq h_{\text{up}}(K) \leq \infty. \tag{4.6}$$

The number $u(0) \stackrel{\Delta}{=} \sup_{\tau \in \mathscr{S}} \mathbb{E}^0[\gamma_0(\tau)B(\tau)]$ is the same as in (3.8); however, unlike the double equality $h_{\text{up}} = h_{\text{low}} = u(0)$ of the unconstrained case, here we have typically $h_{\text{low}}(K) < h_{\text{up}}(K)$. *How does then one characterize, or even compute,* *these upper- and lower-hedging prices in the general, constrained case?* We shall try to address this question in the next sections. For the remainder of the present section, let us take up the important issue of *arbitrage*.

4.2 Definition: *Suppose that* $u > 0$ *is the price of the American Contingent Claim* $B(\cdot)$ *in the market* $\mathcal{M}(K)$*, at time* $t = 0$*. We say that the triple* $(\mathcal{M}(K), u, B(\cdot))$ **admits an arbitrage opportunity**, *if there exists either*

(i) a pair $(\hat{\pi}, \hat{C}) \in \mathcal{A}_+(x)$ *that satisfies*

$$X^{x,\hat{\pi},\hat{C}}(\tau) \geq B(\tau) \text{ a.s.}, \ \forall \tau \in \mathcal{S} \tag{4.7}$$

for some $0 < x < u$*; or*

(ii) a stopping time $\check{\tau} \in \mathcal{S}$ *and a pair* $(\check{\pi}, \check{C}) \in \mathcal{A}_-(-x, \check{\tau})$*, such that*

$$X^{-x,\check{\pi},\check{C}}(\check{\tau}) + B(\check{\tau}) \geq 0, \text{ a.s.} \tag{4.8}$$

holds for some $x > u$*.* □

The *economic meaning* of this definition should be clear. In the first case, an agent can *sell* the contingent claim at time $t = 0$ for $u > x$ (i.e., for more than is required to hedge it without risk throughout the interval $[0, T]$, in the sense of (4.7)). In the second case, an agent can *buy* the contingent claim for $u < x$ (that is, for less than the amount which allows him to recover his initial debt without risk, as in (4.8), by exercising his option to the claim at some stopping time $\check{\tau}$ in $\mathcal{S}$). In either case, there exists an opportunity for creating wealth without risk, i.e., for arbitrage. Clearly, *any price* $u > 0$ *that leads to such an arbitrage opportunity should be excluded*.

4.3 Theorem. *Every* $u > 0$ *outside the interval* $[h_{\text{low}}(K), h_{\text{up}}(K)]$ *leads to an arbitrage opportunity in* $(\mathcal{M}(K), u, B)$*, while no* $u > 0$ *in this interval does. For this reason, we call* $[h_{\text{low}}(K), h_{\text{up}}(K)]$ *the arbitrage-free interval.*

Proof of Theorem 4.3: One checks easily, that the sets $\mathcal{U}, \mathcal{L}$ of (4.4), (4.5), respectively, are intervals: $(x \in \mathcal{L},\ 0 \leq y \leq x) \Rightarrow y \in \mathcal{L}$, and $(x \in \mathcal{U}, y \geq x) \Rightarrow y \in \mathcal{U}$. Now, if $u > h_{\text{up}}(K)$, for any x in the interval $(h_{\text{up}}(K), u)$ we have $x \in \mathcal{U}$, that is (4.7) for some pair $(\hat{\pi}, \hat{C}) \in \mathcal{A}_+(x)$; similarly, if $u < h_{\text{low}}(K)$, for any x in the interval $(u, h_{\text{low}}(K))$ we have $x \in \mathcal{L}$, that is (4.8), for some stopping time $\check{\tau} \in \mathcal{S}$ and some pair $(\check{\pi}, \check{C}) \in \mathcal{A}_-(-x, \check{\tau})$.

Now suppose $h_{\text{low}}(K) \leq u \leq h_{\text{up}}(K)$ and that the conditions of (i) are satisfied; from the definition (4.4) of $h_{\text{up}}(K)$, we obtain then $h_{\text{up}}(K) \leq x < u$, a contradiction. We argue similarly, if the conditions of (ii) are satisfied. □

In the unconstrained setup of Sect. 3, one can show similarly that any price $u \neq u(0)$ leads to an arbitrage opportunity in $(\mathcal{M}, u, B)$.

5. Convex constraints

Let us concentrate henceforth on the important case of *closed, convex* subsets K_+, K_- of $\mathscr{R}^d$. We shall assume, as before, that $K_+ \cap K_-$ contains the origin, but we shall impose also the additional condition

$$\lambda p_+ + (1-\lambda)p_- \in \left\{ \begin{array}{ll} K_+, & \text{if } \lambda \geq 1 \\ K_-, & \text{if } \lambda \leq 0 \end{array} \right\}, \quad \text{for every } p_+ \in K_+,\ p_- \in K_-. \tag{5.1}$$

The function

$$\delta(x) \stackrel{\Delta}{=} \sup_{\pi \in K^+} (-p^* x) : \mathscr{R}^d \to [0, \infty] \tag{5.2}$$

and its *effective domain*

$$\begin{aligned} \tilde{K} &\stackrel{\Delta}{=} \{x \in \mathscr{R}^d / \exists \beta \in \mathscr{R} \text{ s.t. } -p^* x \leq \beta,\ \forall\ p \in K_+\} \\ &= \{x \in \mathscr{R}^d / \delta(x) < \infty\} \end{aligned} \tag{5.3}$$

will play an important role for our subsequent analysis. In the terminology of Convex Analysis (e.g. Rockafellar 1970), $\delta(\cdot)$ is the *support function* of the convex set $-K_+$, and $\tilde{K}$ is a convex cone, called "barrier cone" of the convex set $-K_+$. Condition (5.1) guarantees that the sets $-K_+$ and K_- have the same barrier cone $\tilde{K}$, on which their support functions add up to zero: in other words,

$$\tilde{K} = \{x \in \mathscr{R}^d / \exists \beta \in \mathscr{R} \text{ s.t. } p^* x \leq \beta,\ \forall p \in K_-\} \tag{5.4}$$

and

$$\sup_{p \in K_-} (p^* x) = \left\{ \begin{array}{ll} -\delta(x), & x \in \tilde{K} \\ \infty, & x \notin \tilde{K} \end{array} \right\}. \tag{5.5}$$

5.1 Remark: The reader should consult Karatzas and Kou (1996), Proposition 7.2, for justification of the claims in the last sentence. It is also shown in that paper (Proposition 7.1) that the condition (5.1) guarantees the following "superposition property": for arbitrary but fixed x_1, x_2 in $\mathscr{R}$, and any $(\pi_i, C_i) \in \mathscr{A}(x_i)$ $(i = 1, 2)$, there exists a pair $(\pi, C) \in \mathscr{A}(x_1 + x_2)$ such that

$$X^{x_1+x_2,\pi,C}(t) = X^{x_1,\pi_1,C_1}(t) + X^{x_2,\pi_2,C_2}(t), \quad \forall\ 0 \leq t \leq T. \quad \square \tag{5.6}$$

Finally, we shall assume throughout that

$$\text{the function } \delta(\cdot) \text{ of (5.2) is continuous on } \tilde{K}; \tag{5.7}$$

a sufficient condition for this, is that the barrier cone $\tilde{K}$ of (5.3) be locally simplicial (cf. Rockafellar (1970), Theorem 10.2 on p. 84).

Here are some examples of convex constraint sets that satisfy all the assumptions of this section. In discussing these, it will be useful to recall the number-of-shares processes $\varphi_i(\cdot)$, $i = 0, 1, \ldots, d$ of (2.10).

5.2 Example: Unconstrained case ($\varphi \in \mathscr{R}^{d+1}$). In other words, $K_+ = K_- = \mathscr{R}^d$; then $\tilde{K} = \{0\}$, $\delta(\cdot) = 0$ on $\tilde{K}$.

5.3 Example: Prohibition of short-selling of stocks ($\varphi_i \geq 0,\ 1 \leq i \leq d$). In other words, $K_+ = [0, \infty)^d$, $K_- = (-\infty, 0]^d$; here $\tilde{K} = [0, \infty)^d$, and $\delta(\cdot) \equiv 0$ on $\tilde{K}$.

5.4 Example: Incomplete market ($\varphi_i = 0,\ m+1 \leq i \leq d$). Suppose now that only the first m stocks, $1 \leq m \leq d-1$, can be traded. Then $K_+ = K_- = \{p \in \mathscr{R}^d / p_i = 0,\ \forall i = m+1, \dots, d\}$ and we obtain $\tilde{K} = \{x \in \mathscr{R}^d / x_i = 0,\ \forall i = 1, \dots, m\}$, $\delta(\cdot) \equiv 0$ on $\tilde{K}$.

5.5 Example: Both K_+ and $K_- = -K_+$ are closed convex cones in $\mathscr{R}^d$. Then $\tilde{K} = \{x \in \mathscr{R}^d / p^* x \geq 0,\ \forall p \in K_+\}$, and $\delta(\cdot) \equiv 0$ on $\tilde{K}$. Clearly, this is a generalization of Examples 5.2-5.4.

5.6 Example: Prohibition of borrowing ($\varphi_0 \geq 0$). In other words, $K_+ = \{p \in \mathscr{R}^d / \sum_{i=1}^d p_i \leq 1\}$, $K_- = \{p \in \mathscr{R}^d / \sum_{i=1}^d p_i \geq 1\}$. Then $\tilde{K} = \{x \in \mathscr{R}^d / x_1 = \cdots = x_d \leq 0\}$, and $\delta(x) = -x_1$ on $\tilde{K}$.

5.7 Example: Constraints on borrowing. A generalization of the previous example is $K_+ = \{p \in \mathscr{R}^d / \sum_{i=1}^d p_i \leq k\}$ for some $k > 1$, and $K_- = \{p \in \mathscr{R}^d / \sum_{i=1}^d p_i \geq k\}$. Then $\tilde{K} = \{x \in \mathscr{R}^d / x_1 = \cdots = x_d \leq 0\}$, and $\delta(x) = -kx_1$, for $x \in \tilde{K}$.

5.8 Example: Constraints on the short-selling of stocks. A generalization of Example 5.3 is $K_+ = [-k, \infty)^d$ for some $k > 0$, and $K_- = (-\infty, -k]^d$; then $\tilde{K} = [0, \infty)^d$ and $\delta(x) = k \sum_{i=1}^d x_i$ on $\tilde{K}$.

In the context of European contingent claims, the techniques for handling such convex constraints on portfolio choice were introduced by Cvitanić and Karatzas (1993) and were further extended by Karatzas and Kou (1996). The basic idea is to introduce an *auxiliary family* $\{\mathscr{M}_\nu\}_{\nu \in \mathscr{D}}$ *of random environments*, parametrized by processes $\nu(\cdot)$ in a suitable family $\mathscr{D}$ which contains the market model $\mathscr{M}$ of Sect. 2: $\mathscr{M} = \mathscr{M}_0$, for the choice $\nu \equiv 0$ in $\mathscr{D}$. Within each member $\mathscr{M}_\nu$ of this family, the pricing problem for the American Contingent Claim $B(\cdot)$ is then solved exactly as in Sect. 3; and by analogy with Theorem 3.3, *one obtains*

$$u_\nu(0) \stackrel{\Delta}{=} \sup_{\tau \in \mathscr{S}} \mathbb{E}[\gamma_\nu(\tau) B(\tau)], \quad \nu \in \mathscr{D} \tag{5.8}$$

as both the upper- and lower-hedging prices of $B(\cdot)$ with unconstrained portfolios, in the auxiliary random environment $\mathscr{M}_\nu$. Then, the task is to show that the upper- and lower-hedging prices of $B(\cdot)$, in the constrained market $\mathscr{M}(K)$ of Sect. 4, are given by

$$h_{\text{up}}(K) = \sup_{\nu \in \mathscr{D}} u_\nu(0) = \sup_{\nu \in \mathscr{D}} \sup_{\tau \in \mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau) B(\tau)] =: V, \tag{5.9}$$

$$h_{\text{low}}(K) = \inf_{\nu \in \mathscr{D}} u_\nu(0) = \inf_{\nu \in \mathscr{D}} \sup_{\tau \in \mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau) B(\tau)] =: v, \tag{5.10}$$

respectively (provided, for (5.10), that $V < \infty$ or $v = 0$).

In order to introduce in detail this family $\{\mathscr{M}_\nu\}_{\nu\in\mathscr{D}}$ of random environments, let $\mathscr{H}$ be the space of $\mathbb{F}$-progressively measurable processes $\nu : [0,T]\times\Omega \to \tilde{K}$ which satisfy

$$\mathbb{E}\int_0^T (\|\nu(t)\|^2 + \delta(\nu(t)))dt < \infty. \tag{5.11}$$

For every $\nu \in \mathscr{H}$, consider now the market model $\mathscr{M}_\nu$ as in (2.1), (2.2), but with $r(\cdot), b(\cdot)$ replaced by $r^{(\nu)}(t) \triangleq r(\cdot) + \delta(\nu(\cdot))$ and $b^{(\nu)}(t) \triangleq b(\cdot) + \nu(\cdot) + \delta(\nu(\cdot))\,\mathbf{1}_d$, respectively:

$$dS_0^{(\nu)}(t) = S_0^{(\nu)}(t)(r(t) + \delta(\nu(t)))dt, \quad S_0^{(\nu)}(0) = 1, \tag{5.12}$$

$$dS_i^{(\nu)}(t) = S_i^{(\nu)}(t)\left[\big(b_i(t) + \nu_i(t) + \delta(\nu(t))\big)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t)\right], \tag{5.13}$$

with $S_i^{(\nu)}(0) = s_i \in (0,\infty),\ i = 1,\dots,d$. In this new market model, the analogues of the processes in (2.8) and (2.3)-(2.5) are given as

$$\gamma_\nu(t) \triangleq \frac{1}{S_0^{(\nu)}(t)} = \exp\left[-\int_0^t (r(s) + \delta(\nu(s)))ds\right], \tag{5.14}$$

$$\theta_\nu(t) \triangleq \sigma^{-1}(t)[b^{(\nu)}(t) - r^{(\nu)}(t)\mathbf{1}_d] = \theta(t) + \sigma^{-1}(t)\nu(t), \tag{5.15}$$

$$Z_\nu(t) \triangleq \exp\left[-\int_0^t \theta_\nu^*(s)dW(s) - \frac{1}{2}\int_0^t \|\theta_\nu(s)\|^2 ds\right], \tag{5.16}$$

$$\begin{aligned} W^{(\nu)}(t) &\triangleq W(t) + \int_0^t \theta_\nu(s)ds \\ &= W^{(0)}(t) + \int_0^t \sigma^{-1}(s)\nu(s)ds, \quad 0 \le t \le T, \end{aligned} \tag{5.17}$$

respectively. For every $\nu(\cdot)$ in the subclass

$$\mathscr{D} \triangleq \left\{\nu \in \mathscr{H} \Big/ \sup_{(t,\omega)\in[0,T]\times\Omega} \|\nu(t,\omega)\| < \infty\right\} \tag{5.18}$$

of *bounded* processes in $\mathscr{H}$, the exponential process $Z_\nu(\cdot)$ of (5.16) is a martingale, and the process $W^{(\nu)}(\cdot)$ of (5.17) is a Brownian motion under the *probability measure*

$$\mathbb{P}^\nu(A) \triangleq \mathbb{E}[Z_\nu(T)1_A], \quad A \in \mathscr{F}(T), \tag{5.19}$$

by Girsanov's theorem (Karatzas and Shreve (1991), Sect. 3.5).

5.9 Remark: The equations of (2.2) for the stock-price processes can be written in the form

$$(5.20)\qquad dS_i(t) = S_i(t)\left[(r(t) - \nu_i(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j^{(\nu)}\right], \quad i = 1, \dots, d$$

in terms of the process $W^{(\nu)}(\cdot)$ of (5.17). In the special case of an Incomplete Market (Example 5.4), it develops from this equation that *the discounted stock prices* $\gamma_0(\cdot)S_i(\cdot)$, $i = 1, \dots, m$ *are martingales under each probability measure in the family* $\{\mathbb{P}^\nu\}_{\nu\in\mathscr{D}}$. For this reason, every $\mathbb{P}^\nu$ $(\nu \in \mathscr{D})$ is called "equivalent martingale measure" for the model $\mathscr{M}$ of (2.1)-(2.2), with $i = 1, \dots, m$.

5.10 Remark: Notice that, in conjunction with (5.14) and (5.17), we can rewrite the equation (2.12) for $X(\cdot) \equiv X^{x,\pi,C}(\cdot)$ equivalently as

$$\begin{aligned} &\gamma_\nu(t)X^{x,\pi,C}(t) + \int_{(0,t]} \gamma_\nu(s)dC(s) \\ &\quad + \int_0^t \gamma_\nu(s)X^{x,\pi,C}(s)[\delta(\nu(s)) + \nu^*(s)p(s)]ds \\ (5.21)\qquad &\quad = x + \int_0^t \gamma_\nu(s)X^{x,\pi,C}(s)p^*(s)\sigma(s)dW^{(\nu)}(s), \quad 0 \le t \le T \end{aligned}$$

for every $\nu \in \mathscr{D}$; here again $p(\cdot)$ is the portfolio-proportion process of (2.10).

5.11 Remark: It should be clear now, from the notation of (5.12)-(5.19) and Theorem 3.3, that the quantity $u_\nu(0)$ of (5.8) is indeed the (upper-, and lower-) hedging price of the American Contingent Claim $B(\cdot)$ with unconstrained portfolios in $\mathscr{M}_\nu$, $\forall\, \nu \in \mathscr{D}$; on the other hand, the arguments of Remark 3.2 (in conjunction with the boundedness of $\nu(\cdot)$) show $u_\nu(0) < \infty$, $\forall\, \nu \in \mathscr{D}$. Let us consider also, for every $\nu \in \mathscr{D}$, the analogue

$$(5.22)\qquad \hat{X}_\nu(t) \triangleq \frac{1}{\gamma_\nu(t)} \operatorname{ess\,sup}_{\tau\in\mathscr{S}_{t,T}} \mathbb{E}^\nu[\gamma_\nu(\tau)B(\tau)|\mathscr{F}(t)], \quad 0 \le t \le T,$$

of (3.9). This is the price-process in $\mathscr{M}_\nu$ of the American Contingent Claim $B(\cdot)$; clearly, $\hat{X}_\nu(0) = u_\nu(0)$ and $\hat{X}_\nu(T) = B(T)$, a.s. Finally, we introduce the processes

$$(5.23)\qquad \overline{X}(t) \triangleq \operatorname{ess\,sup}_{\nu\in\mathscr{D}} \hat{X}_\nu(t), \qquad \underline{X}(t) \triangleq \operatorname{ess\,inf}_{\nu\in\mathscr{D}} \hat{X}_\nu(t); \quad 0 \le t \le T.$$

These satisfy $\overline{X}(0) = V$, $\underline{X}(0) = v$ in the notation of (5.9), (5.10), as well as $\overline{X}(T) = \underline{X}(T) = B(T)$, a.s.

Here are the two main results of this paper, which justify the claims of (5.9) and (5.10).

5.12 Theorem: *The upper-hedging price* $h_{\mathrm{up}}(K)$ *of (4.4) is given by*

$$(5.9)\qquad h_{\mathrm{up}}(K) = V \triangleq \sup_{\nu\in\mathscr{D}} u_\nu(0) = \sup_{\nu\in\mathscr{D}} \sup_{\tau\in\mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau)B(\tau)].$$

Furthermore, if $V < \infty$, *there exists a pair* $(\hat{\pi}, \hat{C}) \in \mathscr{A}_+(V)$ *such that*

$$X^{V,\hat{\pi},\hat{C}}(\tau) = \overline{X}(\tau) \geq B(\tau), \quad \forall\, \tau \in \mathscr{S} \tag{5.24}$$

holds, almost surely.

Proof: The inequality $V \leq h_{\text{up}}(K)$ is obvious, if $h_{\text{up}}(K) = \infty$; if not, the set of (4.4) is nonempty. With $x \geq 0$ an arbitrary element of this set, and $(\hat{\pi}, \hat{C}) \in \mathscr{A}_+(x)$ any portfolio/consumption process pair that satisfies (3.2), the process of (5.21) is then a nonnegative local martingale, thus also a supermartingale, under $\mathbb{P}^\nu$. Consequently, from (3.2), (4.2), (5.2) and the Optional Sampling Theorem, we obtain

$$\begin{aligned} x \;&\geq\; \mathbb{E}^\nu\Bigg[\gamma_\nu(\tau)X^{x,\hat{\pi},\hat{C}}(\tau) + \int_{(0,\tau]} \gamma_\nu(t)d\hat{C}(t) \\ &\quad + \int_0^\tau \gamma_\nu(s)X^{x,\hat{\pi},\hat{C}}(s)\{\delta(\nu(s)) + \nu^*(s)\hat{p}(s)\}ds\Bigg] \\ &\geq\; \mathbb{E}^\nu[\gamma_\nu(\tau)B(\tau)],\ \text{ for every } \tau \in \mathscr{S} \text{ and } \nu \in \mathscr{D}, \end{aligned}$$

in the notation of (2.10) for the portfolio-proportion process $\hat{p}(\cdot) = (\hat{p}_1(\cdot), \ldots, \hat{p}_d(\cdot))^*$, namely,

$$\hat{p}_i(t) \overset{\Delta}{=} \left\{ \begin{array}{ll} \frac{\hat{\pi}_i(t)}{X^{x,\hat{\pi},\hat{C}}(t)} & , \text{ if } X^{x,\hat{\pi},\hat{C}}(t) > 0 \\ 0 & , \text{ if } X^{x,\hat{\pi},\hat{C}}(t) = 0 \end{array} \right\}, i = 1, \ldots, d.$$

Therefore, $x \geq \sup_{\nu\in\mathscr{D}} \sup_{\tau\in\mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau)B(\tau)] = V$, and $h_{\text{up}}(K) \geq V$ follows from the arbitrariness of x in the set of (4.4).

Similarly, the inequality $h_{\text{up}}(K) \leq V$ is trivial, if $V = \infty$. In Appendix A we establish this inequality, as well as the remaining claims of the Theorem, for the case $V < \infty$. □

5.13 Theorem: *The lower-hedging price $h_{\text{low}}(K)$ of (4.5) satisfies*

$$h_{\text{low}}(K) \leq v \overset{\Delta}{=} \inf_{\nu\in\mathscr{D}} u_\nu(0) = \inf_{\nu\in\mathscr{D}} \sup_{\tau\in\mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau)B(\tau)], \tag{5.10$'$}$$

with equality if $V < \infty$ or if $v = 0$. In the case $V < \infty$, there exists a pair $(\check{\pi}, \check{C}) \in \mathscr{A}_-(-v)$ such that

$$-X^{-v,\check{\pi},\check{C}}(\cdot \wedge \rho_0) = \underline{X}(\cdot \wedge \rho_0) \geq B(\cdot) \quad \text{and} \quad -X^{-v,\check{\pi},\check{C}}(\rho_0) = B(\rho_0) \tag{5.25}$$

hold almost surely, with

$$\rho_t \overset{\Delta}{=} \inf\{u \in [t,T)/\underline{X}(u) = B(u)\} \wedge T, \quad 0 \leq t \leq T, \tag{5.26}$$

provided that $v > 0$.

Proof: The inequality $v \geq h_{\text{low}}(K)$ is obvious, if $h_{\text{low}}(K) = 0$. If not, the set of (4.5) is nonempty; take an *arbitrary* $x \geq 0$ in this set, as well as any $\check{\tau} \in \mathscr{S}$ and $(\check{\pi}, \check{C}) \in \mathscr{A}_-(-x, \check{\tau})$ for which (3.4) holds. From (5.21), the process

$$\gamma_\nu(t)X^{-x,\check{\pi},\check{C}}(t) + \int_{(0,t]} \gamma_\nu(s)d\check{C}(s)$$

$$+\int_0^t \gamma_\nu(s)X^{-x,\check{\pi},\check{C}}(s)[\delta(\nu(s)) + \nu^*(s)\check{p}(s)]ds, \quad 0 \le t \le T \tag{5.27}$$

is then a $\mathbb{P}^\nu$-local martingale, for every $\nu \in \mathscr{D}$. Here again, we have employed the notation

$$\check{p}_i(t) \stackrel{\Delta}{=} \left\{ \begin{array}{ll} \dfrac{\check{\pi}_i(t)}{X^{-x,\check{\pi},\check{C}}(t)} & , \text{ if } X^{-x,\check{\pi},\check{C}}(t) > 0 \\ 0 & , \text{ if } X^{-x,\check{\pi},\check{C}}(t) = 0 \end{array} \right\}, \quad i = 1, \ldots, d.$$

of (2.10) for the portfolio-proportion process $\check{p}(\cdot) = (\check{p}_1(\cdot), \ldots, \check{p}_d(\cdot))^*$. The second and third terms in this expression are nonnegative (recall here (4.3) and (5.5)), whereas the first term dominates the random variable $-\Lambda \cdot \max_{0\le t\le T} \gamma_\nu(t)$, which is $\mathbb{P}^\nu$-integrable. To see this, take into account the boundedness of $r(\cdot)$, $\nu(\cdot)$ and $\sigma^{-1}(\cdot)$, observe

$$\frac{Z_\nu(t)}{Z_0(t)} = \exp\left[-\int_0^t (\sigma^{-1}(s)\nu(s))^* dW^{(0)}(s) - \frac{1}{2}\int_0^t \|\sigma^{-1}(s)\nu(s)\|^2 ds \right] \tag{5.28}$$

from (5.16), and argue as in Remark 3.2 that

$$\mathbb{E}^\nu(\Lambda) = \mathbb{E}^0\left[\frac{Z_\nu(T)}{Z_0(T)} \cdot \Lambda\right] \le (\mathbb{E}^0(\Lambda^p))^{1/p} \cdot \left(E^0\left(\frac{Z_\nu(T)}{Z_0(T)}\right)^q\right)^{1/q} < \infty \tag{5.29}$$

with $p > 1$ as in Definition 2.4 and $\frac{1}{p} + \frac{1}{q} = 1$.

It develops that the local martingale of (5.27) is bounded from below by a $\mathbb{P}^\nu$-integrable random variable and is thus a supermartingale, under $\mathbb{P}^\nu$. We obtain, from (3.4) and the optional sampling theorem:

$$\begin{aligned} -x &\ge \mathbb{E}^\nu\Big[\gamma_\nu(\check{\tau})X^{-x,\check{\pi},\check{C}}(\check{\tau}) + \int_{(0,\check{\tau}]} \gamma_\nu(t)d\check{C}(t) \\ &\quad + \int_0^{\check{\tau}} \gamma_\nu(t)X^{-x,\check{\pi},\check{C}}(t)[\delta(\nu(t)) + \nu^*(t)\check{p}(t)]dt\Big] \\ &\ge -\mathbb{E}^\nu[\gamma_\nu(\check{\tau})B(\check{\tau})] \ge -u_\nu(0), \quad \forall\, \nu \in \mathscr{D} \end{aligned}$$

whence $v = \inf_{\nu\in\mathscr{D}} u_\nu(0) \ge x$, and thus $v \ge h_{\text{low}}(K)$ from the arbitrariness of $x \ge 0$ in the set of (4.5).

Similarly, the inequality $v \le h_{\text{low}}(K)$ is obvious, if $v = 0$; in Appendix B we establish this inequality, as well as the remaining claims of the theorem, for the case $(v > 0,\ V < \infty)$. □

Let us consider, in addition to those of (5.26), the stopping times

$$\check{\rho}_t(\nu) \stackrel{\Delta}{=} \inf\{u \in [t,T)/\hat{X}_\nu(u) = B(u)\} \wedge T, \quad 0 \le t \le T \tag{5.30}$$

for every $\nu \in \mathscr{D}$, and notice that

$$\rho_t \le \check{\rho}_t(\nu), \quad 0 \le t \le T \quad (\forall\ \nu \in \mathscr{D}). \tag{5.31}$$

In terms of these stopping times we have the following, somewhat simpler, representations for $h_{\mathrm{up}}(K)$ and $h_{\mathrm{low}}(K)$. These are also proved in Appendix B.

5.14 Proposition: *Suppose that $V < \infty$. Then the processes of (5.23) admit the representations*

$$\bar{X}(t) = \operatorname{ess\,sup}_{\nu\in\mathscr{D}} \mathbb{E}^{\nu}\left[\frac{\gamma_\nu(\rho_t)}{\gamma_\nu(t)}\bar{X}(\rho_t)\middle|\mathscr{F}(t)\right], \tag{5.32}$$

$$\underline{X}(t) = \operatorname{ess\,inf}_{\nu\in\mathscr{D}} \mathbb{E}^{\nu}\left[\frac{\gamma_\nu(\rho_t)}{\gamma_\nu(t)}B(\rho_t)\middle|\mathscr{F}(t)\right], \quad 0 \le t \le T. \tag{5.33}$$

In particular, the operations of "infimum" and "supremum" in the definition of $\underline{X}(\cdot)$ can be interchanged:

$$\begin{aligned}\underline{X}(t) &= \operatorname{ess\,inf}_{\nu\in\mathscr{D}} \operatorname{ess\,sup}_{\tau\in\mathscr{S}_{t,T}} \mathbb{E}^{\nu}\left[\frac{\gamma_\nu(\tau)}{\gamma_\nu(t)}B(\tau)\middle|\mathscr{F}(t)\right]\\ &= \operatorname{ess\,sup}_{\tau\in\mathscr{S}_{t,T}} \operatorname{ess\,inf}_{\nu\in\mathscr{D}} \mathbb{E}^{\nu}\left[\frac{\gamma_\nu(\tau)}{\gamma_\nu(t)}B(\tau)\middle|\mathscr{F}(t)\right], \quad 0 \le t \le T\end{aligned} \tag{5.34}$$

and we have

$$h_{\mathrm{low}}(K) = \inf_{\nu\in\mathscr{D}} \mathbb{E}^{\nu}[\gamma_\nu(\rho_0)B(\rho_0)] = \sup_{\tau\in\mathscr{S}} \inf_{\nu\in\mathscr{D}} \mathbb{E}^{\nu}[\gamma_\nu(\tau)B(\tau)], \tag{5.35}$$

$$h_{\mathrm{up}}(K) = \sup_{\nu\in\mathscr{D}} \mathbb{E}^{\nu}[\gamma_\nu(\rho_0)\bar{X}(\rho_0)] = \sup_{\tau\in\mathscr{S}} \sup_{\nu\in\mathscr{D}} \mathbb{E}^{\nu}[\gamma_\nu(\tau)B(\tau)]. \quad \square \tag{5.36}$$

5.15 Remark: In the case of an *American put-option* $B(t) = (q - S_i(t))^+$ with $r(\cdot) \ge 0$, we have clearly $V \le q < \infty$. For an *American call-option* $B(t) = (S_i(t) - q)^+$ we have $V < \infty$ if and only if

$$x \mapsto \delta(x) + x_i \ \text{ is bounded from below on } \tilde{K}. \tag{5.37}$$

This can be shown in exactly the same way as in Remarks 6.8-6.10 of Cvitanić and Karatzas (1993); the condition (5.37) is satisfied, in particular, if

$$\left\{\begin{array}{l} K_+ \text{ contains both the origin and the } i^{\text{th}} \\ \quad \text{unit vector } \ \mathbf{e}_i = (0,\ldots 0,1,0,\ldots,0). \end{array}\right\} \tag{5.37}$$

5.16 Remarks: The reader should not fail to notice that the maximizations in (5.9), (5.23) for $\bar{X}(\cdot)$, involve a *mixed optimal stopping/stochastic control problem*, in which the controller maximizes over both the stopping time $\tau \in \mathscr{S}$ and the control process $\nu \in \mathscr{D}$.

Similarly, the optimization problem in (5.10), (5.23) for $\underline{X}(\cdot)$, involves a *stochastic game* between two players: one of them, the "maximizer", chooses the stopping time $\tau \in \mathscr{S}$, whereas the second player, the "minimizer", gets to choose the process $\nu \in \mathscr{D}$. The order in which these operations are carried out is irrelevant, as (5.34) shows, and thus the game has "value" process $\underline{X}(\cdot)$.

The reduction of the representation (5.10) to (5.35), which involves only the stopping time ρ_0 instead of the entire family $\mathscr{S}$, should not be surprising: it reflects the fact that the buyer has to select just *one* stopping time, which turns out to be ρ_0.

5.17 Remark: In the absence of condition (5.1), the property (5.5) ceases, in general, to be valid; however the representation (5.10) still stands, if one replaces there the class of processes $\mathscr{D}$ by the class $\tilde{\mathscr{D}}$ of Theorem 6.1 in Karatzas and Kou (1996), and (5.9) still holds without any modification.

We should like to point out that there are examples of constraint sets with clear economic meaning, which violate the assumption (5.1). We present two such examples below.

5.18 Example: Constraints on short-selling of stocks. A generalization of Example 5.3 is to take $K_+ = [-k,\ \infty)^d$, $K_- = (-\infty,\ l]^d$, where $k \geq 0$ and $l \geq 0$. Using (2.10), this constraint can be easily translated as

$$\varphi_i(t)S_i(t) = X(t)p_i(t) \geq \left\{ \begin{array}{ll} -kX(t) & ;\text{ if } X(t) > 0 \\ lX(t) & ;\text{ if } X(t) < 0 \end{array} \right\}, \quad i = 1, \ldots, d.$$

In other words, the economic meaning is that the amount of short-selling should be not more than k times the total amount of the wealth, if the wealth is positive, and not more than l times the absolute value of the wealth, if the wealth is negative. Certainly, an interesting special case is $k = l$. Notice that the constraint sets K_+ and K_- increase to $\mathscr{R}^d$ as k and l go to infinity. Intuitively, such a property should lead to the conclusion that the arbitrage-free interval will shrink then to $\{u(0)\}$; the details will be given in the next section.

5.19 Example: Constraints on borrowing. A generalization of Example 5.6 is to take $K_+ = \{p \in \mathscr{R}^d / \sum_{i=1}^d p_i \leq k + 1\}$, $K_- = \{p \in \mathscr{R}^d / \sum_{i=1}^d p_i \geq 1 - l\}$, where $k \geq 0$ and $l \geq 0$. Again using (2.10), it can be translated as

$$\varphi_0(t)S_0(t) = X(t)(1 - \sum_{j=1}^{d} p_j(t)) \geq \left\{ \begin{array}{ll} -kX(t) & ;\text{ if } X(t) > 0 \\ lX(t) & ;\text{ if } X(t) < 0 \end{array} \right\}.$$

In other words, the amount borrowed is limited to not more than k times total wealth, if the wealth is positive, and to l times the absolute value of wealth, if the wealth is negative. Notice again that the constraints become weaker and weaker as k and l increase.

We shall treat these two examples separately in the next section. Their counterparts for European contingent claims are studied in Karatzas and Kou (1996).

We shall close this section with a couple of results about the *American call-option* $B(\cdot) = (S_i(\cdot) - q)^+$ on the i^{th} stock.

5.20 Proposition: *Suppose that we have* $r(\cdot) \geq 0$, *and that for some* $0 \leq \ell < \infty$,

$$-\ell \leq \delta(x) + x_i \leq 0, \quad \forall\, x \in \tilde{K}. \tag{5.38}$$

Then the upper- and lower-hedging prices of (4.4), (4.5) for the American call-option $B(\cdot) = (S_i(\cdot) - q)^+$, are given by their counterparts for the corresponding European call-option

(5.39)
$$h_{\mathrm{up}}(K) = \sup_{\nu \in \mathscr{D}} \mathbb{E}^\nu[\gamma_\nu(T)(S_i(T) - q)^+], \quad h_{\mathrm{low}}(K) = \inf_{\nu \in \mathscr{D}} \mathbb{E}^\nu[\gamma_\nu(T)(S_i(T) - q)^+],$$

respectively.

Proof: From (5.20) and (5.17), (5.14), we have

$$(5.40) \quad d(\gamma_\nu(t)S_i(t)) = (\gamma_\nu(t)S_i(t))\left[-(\delta(\nu(t)) + \nu_i(t))dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j^{(\nu)}(t)\right],$$

or equivalently

$$\gamma_\nu(t)S_i(t) = S_i(0)\exp\left\{ -\int_0^t (\delta(\nu(s)) + \nu_i(s))ds \right\} M_\nu(t)$$
$$(5.41) \quad M_\nu(t) \stackrel{\Delta}{=} \exp\left\{ \int_0^t \sigma_i(s)dW^{(\nu)}(s) - \frac{1}{2}\int_0^t \|\sigma_i(s)\|^2 ds \right\}, \quad 0 \le t \le T.$$

This shows that, under (5.38), the process of (5.41) is a $\mathbb{P}^\nu$-submartingale, for every $\nu \in \mathscr{D}$ (we have introduced the row-vector $\sigma_i(\cdot) = (\sigma_{i1}(\cdot), \ldots, \sigma_{id}(\cdot))$). Consequently, from the decrease of $\gamma_\nu(t) = \exp\{-\int_0^t [r+\delta(\nu(s))]ds\}$ and Jensen's inequality, $\gamma_\nu(\cdot)(S_i(\cdot)-q)^+ = (\gamma_\nu(\cdot)S_i(\cdot)-q\gamma_\nu(\cdot))^+$ is also a $\mathbb{P}^\nu$-submartingale. By analogy with Example 3.6, it develops now again that $u_\nu(0) = \mathbb{E}^\nu[\gamma_\nu(T)(S_i(T) - q)^+]$ in (5.8) ($\forall \nu \in \mathscr{D}$), and (5.39) follows from Theorems 5.12, 5.13 and Remark 5.15.

5.21 Proposition: *Suppose that the interest-rate process $r(\cdot)$ satisfies $r(\cdot) \le r$ for some real constant $r \ge 0$, and that the function*

(5.42) $x \mapsto \delta(x) + x_i$ *is both nonnegative, and unbounded from above, on $\tilde{K}$.*

Then the upper- and lower-hedging prices of (4.4), (4.5) for the American call-option $B(\cdot) = (S_i(\cdot) - q)^+$ satisfy

$$(5.43) \qquad h_{\mathrm{up}}(K) \le S_i(0), \qquad h_{\mathrm{low}}(K) = B(0) = (S_i(0) - q)^+.$$

Proof: The condition (5.42) implies that the process

(5.44) $\gamma_\nu(\cdot)S_i(\cdot)$ of (5.40) is a $\mathbb{P}^\nu-$supermartingale $(\forall \nu \in \mathscr{D})$,

and thus $u_\nu(0) \le \gamma_\nu(0)S_i(0) = S_i(0)$ by the optional sampling theorem; this leads directly to the first claim of (5.43). For the second, observe that given any $0 < \epsilon < T$, $\tau \in \mathscr{S}$ and any element in the space $\mathscr{D}_d$ of bounded, deterministic (nonrandom) functions $\nu : [0, T] \mapsto \tilde{K}$, we have

$$
\begin{aligned}
\mathbb{E}^{\nu}[\gamma_{\nu}(\tau)(S_i(\tau)-q)^{+}] &\le \mathbb{E}^{\nu}[\gamma_{\nu}(\tau)S_i(\tau)1_{\{\tau>\epsilon\}}] \\
&\quad +\mathbb{E}^{\nu}[(\gamma_{\nu}(\tau)S_i(\tau)-qe^{-\int_0^{\tau}(r(s)+\delta(\nu(s)))ds})^{+}\cdot 1_{\{\tau\le\epsilon\}}] \\
&\le \mathbb{E}^{\nu}[\gamma_{\nu}(\epsilon)S_i(\epsilon)]+\mathbb{E}^{\nu}[(S_i(0)M_{\nu}(\tau)-q(\epsilon))^{+}\cdot 1_{\{\tau\le\epsilon\}}] \\
&\le S_i(0)e^{-\int_0^{\epsilon}(\delta(\nu(s))+\nu_i(s))ds}\cdot\mathbb{E}^{\nu}[M_{\nu}(\epsilon)]+\mathbb{E}^{\nu}(S_i(0)M_{\nu}(\epsilon)-q(\epsilon))^{+} \\
&= S_i(0)e^{-\int_0^{\epsilon}(\delta(\nu(s))+\nu_i(s))ds}+A_{\nu}(\epsilon), \qquad (5.45)
\end{aligned}
$$

where $q(\epsilon)\overset{\Delta}{=}q\exp\big(-\int_0^{\epsilon}(r+\delta(\nu(s)))ds\big)$, $M_{\nu}(t)\overset{\Delta}{=}\exp\{\int_0^t\sigma_i(s)dW^{(\nu)}(s)-\frac{1}{2}\int_0^t\|\sigma_i(s)\|^2ds\}$ is the $\mathbb{P}^{\nu}$-martingale that appears in (5.41), and $A_{\nu}(\epsilon)\overset{\Delta}{=}\mathbb{E}^{\nu}(S_i(0)M_{\nu}(\epsilon)-q(\epsilon))^{+}$. In deriving (5.45) we have used the property (5.44), the nonnegativity of $x\mapsto\delta(x)+x_i$ on $\tilde{K}$, and Jensen's inequality (to argue that $(S_i(0)M_{\nu}(\cdot)-q(\epsilon))^{+}$ is a $\mathbb{P}^{\nu}$-submartingale). Now (5.45), (5.42) and Theorem 5.13 give

$$
\begin{aligned}
h_{\text{low}}(K) &\le \inf_{\nu\in\mathscr{D}_d}\sup_{\tau\in\mathscr{S}}\mathbb{E}^{\nu}[\gamma_{\nu}(\tau)(S_i(\tau)-q)^{+}] \\
&\le A_0(\epsilon)+S_i(0)\cdot\inf_{\nu\in\mathscr{D}_d}e^{-\int_0^{\epsilon}(\delta(\nu(s))+\nu_i(s))ds}=A_0(\epsilon)
\end{aligned}
$$

for every $0<\epsilon<T$. But the family of random variables $\{M_0(\epsilon)\}_{0<\epsilon<T}$ is uniformly integrable under $\mathbb{P}^0$, as is checked by observing that $\sup_{0<\epsilon<T}\mathbb{E}^0(M_0(\epsilon))^2\le\exp(kT)$ holds, where k is an upper bound on $\|\sigma_i(\cdot)\|^2$. Thus, we have $h_{\text{low}}(K)\le\overline{\lim}_{\epsilon\downarrow 0}A_0(\epsilon)=(S_i(0)-q)^{+}=B(0)$. The reverse inequality is already in (4.6). □

6. Market with constant coefficients

Let us consider now the case of a market with *constant coefficients*

$$
r(\cdot)\equiv r\ge 0,\quad \sigma(\cdot)\equiv\sigma, \qquad (6.1)
$$

where σ is a given, invertible, $(d\times d)$ matrix, and a payoff process of the form

$$
B(t)=\varphi(S_1(t),\dots,S_d(t)),\quad 0\le t\le T. \qquad (6.2)
$$

Here $\varphi:(0,\infty)^d\to[0,\infty)$ is a continuous function, and we assume that the condition (3.1) is satisfied.

With given constraint sets K_+ and K_-, as in the beginning of Sect. 5, we define the functions

$$
\overline{\varphi}(x)\overset{\Delta}{=}\sup_{\nu\in\tilde{K}}\left[e^{-\delta(\nu)}\varphi\left(x_1e^{-\nu_1},\dots,x_de^{-\nu_d}\right)\right],\quad x\in(0,\infty)^d \qquad (6.3)
$$

$$
\underline{\varphi}(x)\overset{\Delta}{=}\inf_{\nu\in\tilde{K}}\left[e^{-\delta(\nu)}\varphi\left(x_1e^{-\nu_1},\dots,x_de^{-\nu_d}\right)\right],\quad x\in(0,\infty)^d\,. \qquad (6.4)
$$

Our next result shows that, under the conditions (6.1) and (6.2), the mixed optimal-stopping/stochastic-control problem of (5.9) can be reduced to a *pure* optimal stopping problem. This reduction is along the lines of a similar result for European contingent claims, due to Broadie et al. (1996).

6.1 Theorem: *The quantity of Theorem 5.12, namely*

$$(5.9)\qquad V \triangleq \sup_{\nu\in\mathscr{D}} \sup_{\tau\in\mathscr{S}} \mathbb{E}^{\nu}\left[e^{-r\tau-\int_0^{\tau}\delta(\nu(s))ds}\cdot\varphi\left(S_1(\tau),\ldots,S_d(\tau)\right)\right] = h_{\text{up}}(K),$$

is given by

$$(6.5)\qquad V = \sup_{\tau\in\mathscr{S}} \mathbb{E}^{0}\left[e^{-r\tau}\cdot\bar{\varphi}\left(S_1(\tau),\ldots,S_d(\tau)\right)\right]$$

under the conditions (6.1), (6.2) and in the notation of (6.3).

A slightly weaker result holds for the lower hedging price.

6.2 Theorem. *The quantity of Theorem 5.13, namely*

$$(5.10)'\qquad v \triangleq \inf_{\nu\in\mathscr{D}} \sup_{\tau\in\mathscr{S}} \mathbb{E}^{\nu}\left[e^{-r\tau-\int_0^{\tau}\delta(\nu(s))ds}\cdot\varphi\left(S_1(\tau),\ldots,S_d(\tau)\right)\right] \geq h_{\text{low}}(K),$$

satisfies

$$(6.6)\qquad v \geq \sup_{\tau\in\mathscr{S}} \mathbb{E}^{0}\left[e^{-r\tau}\cdot\underline{\varphi}\left(S_1(\tau),\ldots,S_d(\tau)\right)\right]$$

under the conditions (6.1), (6.2) and in the notation of (6.4).

Recall from Theorem 5.13 that (5.10)$'$ holds as equality, if $v = 0$ or if $V < \infty$. Clearly (6.6) also holds as equality if $v = 0$; as we show in (6.13) below, the inequality (6.6) may be strict if $v > 0$, even when $V < \infty$ holds.

Both Theorems 6.1 and 6.2 will be proved in Appendix C. For the reminder of this section, let us discuss the case of an *American call-option*

$$(6.7)\qquad B(t) = \left(S_i(t) - q\right)^{+},\quad 0 \leq t \leq T$$

on the i^{th} stock, with exercise price $q \geq 0$, as in Example 3.6, but now under various constraints on portfolio choice. We place ourselves under condition (6.1), except in the discussion of Examples 5.18 and 5.19.

6.3 Remark: Under the assumption (5.42), we have now from Theorem 6.1 the representation

$$(6.8)\qquad h_{\text{up}}(K) = \sup_{\tau\in\mathscr{S}} \mathbb{E}^{0}\left[e^{-r\tau}\cdot\bar{\varphi}\left(S_i(\tau)\right)\right]$$

for the upper-hedging-price in (5.43), where

$$(6.9)\qquad \bar{\varphi}(x) = \sup_{\nu\in\tilde{K}}\left[e^{-\delta(\nu)}\left(xe^{-\nu_i} - q\right)^{+}\right],\quad x\in(0,\infty)$$

is the function of (6.3).

5.6 Example: Prohibition of borrowing (continued). In this case we have $\delta(x) + x_i = 0$ $(\forall x \in \tilde{K})$ and the quantities of (5.39) are given by

$$(6.10) \qquad h_{\text{up}}(K) = S_1(0), \quad \text{and} \quad h_{\text{low}}(K) = u(0) = \mathbb{E}^0[e^{-rT}(S_i(T) - q)^+]$$

as in (3.27)-(3.30), respectively; see Example 7.2 in Cvitanić and Karatzas (1993), as well as Example 8.1 and Remark 8.1 in Karatzas and Kou (1996).

5.4 Example: Incomplete market, $1 \leq i \leq m$ (continued). Here, too, we have $\delta(x) + x_i = 0$ $(\forall x \in \tilde{K})$, and (6.2) gives

$$h_{\text{up}}(K) = h_{\text{low}}(K) = u(0) = \mathbb{E}^0[e^{-rT}(S_i(T) - q)^+];$$

recall Example 8.5 (a) in Karatzas and Kou (1996).

5.3 Example: Prohibition of short-selling of stocks (continued). In this case $\delta(x) + x_i = x_i$ is both nonnegative and unbounded on $\tilde{K} = [0, \infty)^d$ and we have

$$(6.11) \quad h_{\text{low}}(K) = (S_1(0) - q)^+ = v, \quad h_{\text{up}}(K) = \mathbb{E}^0[e^{-rT}(S_i(T) - q)^+] = V = u(0)$$

as in Examples 1.4.7, 1.3.2 in Karatzas (1997). Indeed, the first of these claims follows from (5.43) of Proposition 5.21. For the second claim, recall from these Examples (loc. cit.) that the portfolio-proportion process $\hat{p}(\cdot)$ takes values in $K_+ = [0, \infty)^d$ and that $u(0)$ belongs to the set of (4.4), since $\hat{X}_0(t) = X^{u(0), \hat{\pi}, 0}(t) = \mathbb{E}^0[e^{-r(T-t)}(S_i(T) - q)^+ | \mathscr{F}(t)] \geq (S_i(t) - q)^+ = B(t),\ 0 \leq t \leq T$ holds almost surely; this gives $h_{\text{up}}(K) \leq u(0)$, whereas the reverse inequality comes from (4.6).

Notice that in this case, and with $d = 1$ for simplicity, the function of (6.4) is given by

$$(6.12) \qquad \underline{\varphi}(x) = \inf_{\nu \geq 0} \left(xe^{-\nu} - q\right)^+ = 0, \quad 0 < x < \infty$$

and thus, for $S_1(0) > q$, the inequality of (6.6) is strict:

$$(6.13) \qquad v = S_1(0) - q > 0 = \sup_{\tau \in \mathscr{S}} \mathbb{E}^0\left[e^{-r\tau} \cdot \underline{\varphi}(S_1(\tau))\right].$$

□

5.8 Example: Constraints on the short-selling of stocks (continued). Here again, $\delta(x) + x_i = (1 + k)x_i + k \sum_{j \neq i} x_j$ is nonnegative and unbounded on $\tilde{K} = [0, \infty)^d$; the same argument as before leads again to the formulae of (6.11).

5.7 Example: Constraints on borrowing (continued). In this case $\delta(x) + x_i = (1 - k)x_i$ is nonnegative and unbounded on $\tilde{K} = \{x \in \mathscr{R}^d / x_1 = \cdots = x_d \leq 0\}$, since $k > 1$; therefore, the lower hedging price is

$$(6.14) \qquad h_{\text{low}}(K) = (S_i(0) - q)^+ = B(0),$$

from Proposition 5.21. On the other hand, we have

(6.15)
$$u(0) \le h_{\rm up}(K) \le a_k \triangleq \frac{k-1}{k}\mathbb{E}^0\left[e^{-rT}\left(S_i(T) - \frac{kq}{k-1}\right)^+\right] + \frac{S_i(0)}{k} \le S_i(0)$$

for the upper hedging price. To see this, recall from Karatzas and Kou (1996) (Example 8.1, proof of (8.2)) that there exists a portfolio $\tilde{\pi}(\cdot)$ with corresponding wealth-process given by

$$\tilde{X}(t) \triangleq X^{a_k,\tilde{\pi},0}(t) = \mathbb{E}^0\left[e^{-r(T-t)}\left(\frac{k-1}{k}S_i(T) - q\right)^+\Big|\mathscr{F}(t)\right] + \frac{S_i(t)}{k},\ 0 \le t \le T$$

and with portfolio-proportion process $\tilde{p}_i(\cdot) \triangleq \tilde{\pi}(\cdot)/\tilde{X}(\cdot)$ satisfying $\tilde{p}_i(\cdot) \le k$ and $\tilde{p}_j(\cdot) = 0,\ \forall\, j \ne i$ (i.e. $\tilde{p}(\cdot) \in K$). Using the fact that $e^{-rt}S_i(t)$ is a $\mathbb{P}^0$-martingale, and thus $e^{-rt}(S_i(t) - q)^+$ a $\mathbb{P}^0$-submartingale, we deduce

$$\begin{aligned}\tilde{X}(t) &= \mathbb{E}^0\left[e^{-r(T-t)}\left\{\left(\frac{k-1}{k}S_i(T) - q\right)^+ + \frac{S_i(T)}{k}\right\}\Big|\mathscr{F}(t)\right] \\ &\ge \mathbb{E}^0[e^{-r(T-t)}(S_i(T) - q)^+|\mathscr{F}(t)] \ge (S_i(t) - q)^+ = B(t), \quad \forall\, 0 \le t \le T\end{aligned}$$

almost surely. In other words, a_k belongs to the set of (4.4), thus $h_{\rm up}(K) \le a_k$. Notice that, as k increases, the bounds of (6.15) become tighter, and $a_k \searrow u(0)$ as $k \to \infty$.

Note that, in this case, the function of (6.9) becomes

(6.16)
$$\bar{\varphi}(x) = \left\{\begin{array}{ll}(c-q)\left(x/c\right)^k & ;\ 0 < x \le c \\ x - q & ;\ x > c\end{array}\right\},\quad c = \frac{kq}{k-1},$$

and thus the upper hedging price $h_{\rm up}(K)$ is given as the optimal expected payoff of the optimal stopping problem (6.8) for this reward function. We have not been able to solve this optimal problem explicitly; see, however, (7.22)-(7.24) below for the solution of this problem on an *infinite time-horizon*. □

5.4 Example: Incomplete market, $m + 1 \le i \le d$ (continued). In this case the stock, on which the American call-option $B(\cdot) = (S_i(\cdot) - q)^+$ is written, cannot be traded. We have that $\delta(x) + x_i = x_i$ is unbounded on $\tilde{K} = \{x \in \mathscr{R}^d / x_j = 0,\ \forall\, j = 1, \ldots, m\}$, both from above and from below; thus $V \triangleq \sup_{\nu\in\mathscr{D}} u_\nu(0) = \infty$ from Remark 5.15, and

(6.17)
$$h_{\rm up}(K) = \infty$$

from Theorem 5.12. On the other hand, arguments similar to those in the proof of Proposition 6.2 give

$$v \triangleq \inf_{\nu\in\mathscr{D}} u_\nu(0) \le (S_i(0) - q)^+ = B(0).$$

Indeed, we have for any $\tau \in \mathscr{S}$, $0 < \epsilon < T$, $\nu \in \mathscr{D}_d$, in the notation of (5.41) and (5.45), that

$$
\begin{aligned}
&\mathbb{E}^{\nu}[\gamma_{\nu}(\tau)(S_i(\tau)-q)^+] \\
&\quad \le \mathbb{E}^{\nu}[\big(S_i(0)\exp\{\int_0^T \nu_i^-(s)ds - \int_0^{\epsilon} \nu_i^+(s)ds\}\cdot M_{\nu}(\tau) - qe^{-rT}\big)^+ 1_{\{\tau>\epsilon\}}] \\
&\qquad\qquad +\mathbb{E}^{\nu}[(S_i(0)\exp\{\int_0^{\epsilon} \nu_i^-(s)ds\}\cdot M_{\nu}(\tau) - qe^{-r\epsilon})^+ 1_{\{\tau\le\epsilon\}}] \\
&\quad \le S_i(0)e^{\int_0^T \nu_i^-(s)ds - \int_0^{\epsilon}\nu_i^+(s)ds} + A_{\nu}(\epsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{\nu}(\epsilon) \;\stackrel{\Delta}{=}\; &\int_{\mathscr{R}} \Bigg(S_i(0)\exp\left\{\int_0^{\epsilon} \nu_i^-(s)ds\right. \\
&\left. +\|\sigma_i\|z\sqrt{\epsilon} - \frac{1}{2}\|\sigma_i\|^2\epsilon\right\} - qe^{-r\epsilon}\Bigg)^+ \cdot \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz\,,
\end{aligned}
$$

and we have denoted $\|\sigma_i\| \stackrel{\Delta}{=} \sqrt{\sum_{j=1}^d \sigma_{ij}^2}$. Taking the infimum over $\nu \in \mathscr{D}_d$, we obtain

$$
v \le \inf_{\nu\in\mathscr{D}_d}\sup_{\tau\in\mathscr{S}} \mathbb{E}^{\nu}[\gamma_{\nu}(\tau)(S_i(\tau)-q)^+] \le A_0(\epsilon), \qquad \forall\; 0<\epsilon<T,
$$

and thus $v \le \overline{\lim}_{\epsilon\downarrow 0} A_0(\epsilon) = (S_i(0)-q)^+$. This way we deduce $h_{\text{low}}(K) \le (S_i(0)-q)^+ = B(0)$. The reverse inequality is also valid, thanks to (4.6), and gives

$$
h_{\text{low}}(K) = (S_i(0)-q)^+ = B(0). \tag{6.18}
$$

5.18 Example: Constraints on the short-selling of stocks (continued). Arguing as in Example 5.3, we have directly

$$
h_{\text{up}}(K) = \mathbb{E}^0[e^{-rT}(S_i(T)-q)^+] = u(0)
$$

by (3.27)-(3.29). Furthermore, the price of an American option must be higher than that of its European counterpart, and thus we get from Example 8.2 and Remark 8.3 in Karatzas and Kou (1996) that

$$
u(0) \ge h_{\text{low}}(K) \ge \rho_l > 0,
$$

for $l>1$; here $\rho_l \stackrel{\Delta}{=} \mathbb{E}^0[e^{-rT}(S_1(T)-q)^+\mathbf{1}_{\{S_1(T)\ge \frac{ql}{l-1}\}}]$ can be computed explicitly as in Remark 8.3 or Sect. 10 of Karatzas and Kou (1996), and $\rho_l \nearrow u(0)$ as $l\to\infty$.

5.19 Example: Constraints on borrowing (continued). From (3.27)-(3.30), the portfolio-proportion process $\hat{p}(\cdot)$ takes values in K_- and $u(0)$ belongs to the set of (4.5). Thus, $h_{\text{low}}(K) = u(0)$, and the previous analysis of Example 5.7 yields $u(0) \le h_{\text{up}}(K) \le a_k$.

7. American call-option on infinite-horizon and with dividends

We shall consider in this section the *American call-option* $B(t) = (S(t) - q)^+$, $0 \le t < \infty$ *on an infinite horizon* in a market model with constant coefficients $r > 0$, $\sigma = \sigma_{11} > 0$, $q > 0$, and one stock ($d = 1, S(\cdot) = S_1(\cdot)$) which pays *dividends* at a certain fixed rate $\beta \in (0, r)$.

This constant dividend rate makes itself felt in the wealth-equation (2.9), where one should replace $b_1(\cdot)$ by $b_1(\cdot) + \beta$; equivalently, this means that the relative-risk process of (2.3) becomes

$$\theta(t) \triangleq \frac{1}{\sigma}(b_1(t) + \beta - r), \quad 0 \le t < \infty, \tag{7.1}$$

and (2.4)-(2.6) are to be understood now with the new definition (7.1) of $\theta(\cdot)$.

7.1 Remark: The fact that we are working now on the infinite time-horizon $[0, \infty)$, rather than on a finite time-interval $[0, T]$, necessitates certain changes in the measure-theoretic setup of the model, particularly concerning the measurability requirements on the processes $b_1(\cdot)$, $\pi(\cdot)$, $C(\cdot)$, $\nu(\cdot)$ and the construction of the probability measures $\mathbb{P}^0$, $\mathbb{P}^\nu$ ($\nu \in \mathscr{D}$). These can be taken care of as in Sect. 1.7 of Karatzas and Shreve (1998), where we refer the reader for details.

Denoting by $x = S(0) \in (0, \infty)$ the initial stock-price, we have now from (2.2), (7.1), and (2.5):

$$S(t) = x \cdot \exp[\sigma W^{(0)}(t) + (r - \beta - \frac{\sigma^2}{2})t] = x \cdot \exp[\sigma(W^{(0)}(t) - \rho t)] \quad 0 \le t < \infty \tag{7.2}$$

where $\rho \triangleq \frac{\beta - r}{\sigma} + \frac{\sigma}{2}$. From this, it is not hard to verify the properties

$$\mathbb{E}^0\left[\sup_{0 \le t \le \infty} (Y(t))^{1+\epsilon}\right] < \infty, \quad \text{for } 0 < \epsilon < \frac{2\beta}{\sigma^2} \tag{7.3}$$

as well as $\lim_{t\to\infty} Y(t) = 0$ (a.s. $\mathbb{P}^0$) for the discounted process

$$Y(t) \triangleq \left\{ \begin{matrix} e^{-rt}(S(t) - q)^+; & 0 \le t < \infty \\ 0; & t = 0 \end{matrix} \right\}.$$

In particular, the condition (3.1) is satisfied.

Let us deal first with the *unconstrained case*; this is a well-known problem, going back to McKean's (1965) classic paper, but we shall sketch here the main lines of the argument for completeness and future reference; all the details can be found in Sect. 2.6 of Karatzas and Shreve (1998). Our main effort will go into computing the *optimal reward function*

$$G(x) \triangleq \sup_{\tau \in \mathscr{S}_{0,\infty}} \mathbb{E}^0[e^{-r\tau}(S(\tau) - q)^+], \quad 0 < x < \infty \tag{7.4}$$

for a given $q \in (0, \infty)$, because then the *price-process* of (3.9) becomes

$$\begin{aligned}\hat{X}_0(t) &= \text{ess sup}_{\tau\in\mathcal{S}_{t,\infty}}\mathbb{E}^0[e^{-r(\tau-t)}(S(\tau)-q)^+] \\ &= G(S(t)) \geq (S(t)-q)^+ = B(t), \quad 0\leq t<\infty, \end{aligned} \tag{7.5}$$

and Itô's rule gives the *optimal hedging portfolio* $\hat{\pi}(\cdot)$ of Theorem 3.3 as

$$\begin{aligned}\hat{\pi}(t) &= S(t)\,G'(S(t)), \\ \hat{p}(t) &= \frac{\hat{\pi}(t)}{\hat{X}_0(t)} = \frac{S(t)\,G'(S(t))}{G(S(t))}, \quad 0\leq t<\infty. \end{aligned} \tag{7.6}$$

In order to compute the optimal reward function of (7.4), we look first at stopping times of the form

$$\begin{aligned}\tau_a &\triangleq \inf\{t\geq 0/S(t)\geq a\} \\ &= \inf\left\{t\geq 0\Big/W^{(0)}(t)-\rho t\geq \frac{1}{\sigma}\log\left(\frac{a}{x}\right)\right\}, \quad a\in(q,\infty). \end{aligned} \tag{7.7}$$

We shall find in this class a stopping time τ_b that maximizes the expected discounted reward $\mathbb{E}^0[e^{-r\tau}(S(\tau)-q)^+]$, and then argue that τ_b is optimal among *all* stopping times. Standard theory (e.g. Karatzas and Shreve (1991), p.176) gives the Laplace transform of the distribution of the stopping time τ_a in (7.7), as

$$\mathbb{E}^0(e^{-r\tau_a}) = \left(\frac{x}{a}\right)^{\gamma}, \quad \text{where } \gamma \triangleq \frac{\sqrt{\rho^2+2r}+\rho}{\sigma}\in\left(1,\frac{r}{r-\beta}\right) \tag{7.8}$$

is the positive root of the equation $\frac{\sigma}{2}\xi^2-\rho\xi-\frac{r}{\sigma}=0$. Thus,

$$g_a(x)\triangleq E^0[e^{-r\tau_a}(S(\tau_a)-q)^+] = \left\{\begin{array}{ll}(a-q)(\frac{x}{a})^{\gamma}; & 0<x<a \\ x-q; & a\leq x<\infty\end{array}\right\} \tag{7.9}$$

for every $a>q$, and

$$g(x)\triangleq \sup_{a>q} g_a(x) = g_b(x), \quad 0<x<\infty, \quad \text{where } b\triangleq\frac{\gamma q}{\gamma-1}. \tag{7.10}$$

It follows from (7.9), (7.10) that the function $g_a(\cdot)$ is convex increasing with

$$0\leq g_a'(\cdot)\leq 1 \tag{7.11}$$

and of class $C^1((0,\infty)\backslash\{a\})$, for every $a\in(q,\infty)$; it is of class $C^1(0,\infty)$, i.e. we have the *"smooth-fit" condition*

$$g_a'(a-)=1=g_a'(a+), \quad \text{if and only if} \quad a=b. \tag{7.12}$$

In fact, the function $g(\cdot)=g_b(\cdot)$ of (7.10) is of class $C^1(0,\infty)\cap C^2((0,\infty)\backslash\{b\})$, and solves the *variational inequality*

$$\begin{array}{rll}\frac{\sigma^2}{2}x^2g''(x)+(r-\beta)xg'(x)-rg(x) & =0; & 0<x<b \\ & =-(\beta x-rq)<0; & x>b \\ g(x) & >(x-q)^+; & 0<x<b \\ g(x) & =x-q; & x\geq b.\end{array} \tag{7.13}$$

7.2 Theorem: McKean (1965). *The optimal reward function $G(\cdot)$ of (7.4) is given by*

$$G(x) = g(x) = g_b(x) = \left\{ \begin{array}{ll} (b-q)(\frac{x}{b})^\gamma; & 0 < x < b \\ x - q; & b \le x < \infty \end{array} \right\}, \quad b = \frac{\gamma q}{\gamma - 1} \tag{7.14}$$

as in (7.8)-(7.10). In terms of it, the price-process $\hat{X}_0(\cdot)$ of (3.9), the optimal hedging portfolio-proportion process $\hat{p}(\cdot)$, the optimal exercise time $\check{\tau}$ of (3.10) and the cash-flow process $\hat{C}(\cdot)$ of Theorem 3.3, are given by (7.5),

$$\hat{p}(t) = \left\{ \begin{array}{ll} \gamma & ; \quad 0 < S(t) < b \\ \frac{S(t)}{S(t)-q} & ; \quad b \le S(t) < \infty \end{array} \right\} \in (1, \gamma], \quad 0 \le t < \infty \tag{7.15}$$

$$\check{\tau} = \tau_b \stackrel{\Delta}{=} \inf\{t \ge 0 / S(t) \ge b\}, \qquad \text{and} \tag{7.16}$$

$$\hat{C}(t) = \int_0^t (\beta S(u) - rq) 1_{\{S(u) > b\}} du, \quad 0 \le t < \infty \tag{7.17}$$

respectively. In particular,

$$\begin{aligned} u(0) &= \hat{X}(0) = G(S(0)), \\ \hat{X}_0(t) &= X^{u(0),\hat{\pi},\hat{C}}(t) = G(S(t)) \ge (S(t) - q)^+ = B(t), \quad 0 \le t < \infty, \\ \hat{C}(\check{\tau}) &= 0, \quad X^{u(0),\hat{\pi},\hat{C}}(\check{\tau}) = B(\check{\tau}). \end{aligned} \tag{7.18}$$

Let us deal now with *convex constraints* on portfolio, as in Sect. 5. We have the analogues of (5.20) and (5.41)

$$\begin{aligned} dS(t) &= S(t)[(r - \beta - \nu(t))dt + \sigma dW^{(\nu)}(t)], \\ \gamma_\nu(t) S(t) &= S(0) \exp\left[- \int_0^t (\beta + \delta(\nu(u)) + \nu(u)) du + \sigma W^{(\nu)}(t) - \frac{\sigma^2}{2} t \right], \end{aligned} \tag{7.19}$$

for $0 \le t < \infty$, and it is relatively easy to check that *Proposition 5.21 remains valid* in this infinite-horizon case as well.

5.3 Example: Prohibition of short-selling of stock (continued). Here $K_+ = [0, \infty)$, $K_- = (-\infty, 0]$ and $\delta(x) + x = x$ is both nonnegative and unbounded from above on $\tilde{K} = [0, \infty)$, which leads to

$$h_{\text{low}}(K) = B(0) = (S(0) - q)^+, \quad h_{\text{up}}(K) = u(0) = G(S(0)). \tag{7.20}$$

Indeed, the first claim follows from Proposition 6.2, whereas $h_{\text{up}}(K) \le u(0) = G(S(0))$ is a consequence of (7.15) (which implies $\hat{p}(\cdot) \in K_+$) and of (7.5), (7.18) (which imply then that $u(0)$ belongs to the set of (4.4)); the reverse inequality is a consequence of (4.6).

5.8 Example: Constraints on the short-selling of stock (continued). In this case $K_+ = [-k, \infty)$, $K_- = (-\infty, -k]$ for some $k > 0$, and again $\delta(x) + x = (1+k)x$ is nonnegative and unbounded from above on $\tilde{K} = [0, \infty)$; arguments similar to those of the previous case lead to the same computations as in (7.20).

5.7 Example: Constraints on borrowing (continued). For some $k > 1$, consider $K_+ = (-\infty, k]$, $K_- = [k, \infty)$ and thus $\delta(x) + x = (1-k)x \geq 0$ on $\tilde{K} = (-\infty, 0]$. From Proposition 6.2, we conclude

$$h_{\text{low}}(K) = B(0) = (S(0) - q)^+, \tag{7.21}$$

and from Theorem 7.2:

$$h_{\text{up}}(K) = u(0) = G(S(0)), \quad \text{if } k \geq \gamma. \tag{7.22}$$

We claim that

$$h_{\text{up}}(K) = G_k(S(0)) \quad \text{for } 1 < k < \gamma, \tag{7.23}$$

where

$$G_k(x) \triangleq \begin{Bmatrix} (c-q)(\frac{x}{c})^k & ; 0 < x < c \\ x - q & ; c \leq x < \infty \end{Bmatrix} \quad \text{and} \quad c \triangleq \frac{kq}{k-1} > b. \tag{7.24}$$

Notice that this $G_k(\cdot)$ is a convex increasing function, of class $C^1(0, \infty)$ ("smooth-fit") and $C^2((0, \infty) \setminus \{c\})$. We shall denote by $G_k''(\cdot)$ the right-hand second derivative of this function on $(0, \infty)$.

Proof of (7.23): $h_{\text{up}}(K) \leq G_k(S(0))$. To prove this inequality it suffices to show that $G_k(S(0))$ belongs to the set of (4.4); that is, to construct a portfolio $\hat{\pi}_k(\cdot)$ and a cumulative consumption process $\hat{C}_k(\cdot)$, such that

$$\hat{X}(t) \triangleq X^{G_k(S(0)), \hat{\pi}_k, \hat{C}_k}(\cdot) \geq B(\cdot) = (S(\cdot) - q)^+, \quad \hat{p}(\cdot) = \frac{\hat{\pi}(\cdot)}{\hat{X}(\cdot)} \in K \text{ a.s.} \tag{7.25}$$

In order to do this, we apply the change-of-variable formula to the process $Y_k(t) \triangleq e^{-rt} G_k(S(t)), \quad 0 \leq t < \infty$, and obtain

$$\begin{aligned} dY_k(t) &= e^{-rt}\left(\frac{\sigma^2}{2}x^2 G_k''(x) + (r - \beta)x G_k'(x) - r G_k(x)\right)\Bigg|_{x=S(t)} dt \\ &\quad + \sigma Y_k(t) \frac{x G_k'(x)}{G_k(x)}\Bigg|_{x=S(t)} dW^{(0)}(t) \end{aligned} \tag{7.26}$$

in conjunction with (7.2). The observations

$$\frac{x G_k'(x)}{G_k(x)} = \begin{Bmatrix} k; & 0 < x < c \\ \frac{x}{x-q}; & x \geq c \end{Bmatrix} \leq k,$$

and

$$-\left[\frac{\sigma^2}{2}x^2 G_k''(x) + (r-\beta)xG_k'(x) - rG_k(x)\right]$$
$$= \left\{ \begin{array}{ll} \beta x - rq \geq \beta c - rq \geq \beta b - rq > 0; & x \geq c \\ -\sigma\left(\frac{\sigma}{2}k^2 - \rho k - \frac{r}{\sigma}\right)G_k(x) > 0; & 0 < x < c \end{array} \right\}$$

show that $\hat{p}_k(t) \stackrel{\Delta}{=} \frac{xG_k'(x)}{G_k(x)}\Big|_{x=S(t)}$ is a portfolio process with values in K_+, and that

$$\hat{C}_k(t) \stackrel{\Delta}{=} -\int_0^t \left(\frac{\sigma^2}{2}x^2 G_k''(x) + (r-\beta)xG_k'(x) - rG_k(x)\right)\Big|_{x=S(u)} du$$

is an (increasing) cumulative consumption process. Back into (7.26), we conclude

$$\hat{X}(t) = X^{G_k(S(0)),\hat{\pi}_k,\hat{C}_k}(t) = G_k(S(t)) \geq (S(t)-q)^+, \qquad 0 \leq t < \infty,$$

and (7.25) follows.

Proof of (7.23): $h_{\text{up}}(K) \geq G_k(S(0))$. For any constant $\nu \in \tilde{K} = (-\infty, 0]$ we have from (5.9) that

$$h_{\text{up}}(K) \geq \sup_{\tau \in \mathscr{S}_{0,\infty}} \mathbb{E}^\nu[e^{-(r-k\nu)\tau}(S(\tau)-q)^+] =: G_k^{(\nu)}(S(0)). \tag{7.27}$$

The optimal stopping problem of (7.27) can be solved explicitly, and exactly as in Theorem 7.2; its optimal expected reward function is given by the analogue

$$G_k^{(\nu)}(x) = \left\{ \begin{array}{ll} (c_\nu - q)(\frac{x}{c_\nu})^{\gamma_\nu}; & 0 < x < c_\nu \\ x - q; & c_\nu \leq x < \infty \end{array} \right\} \tag{7.28}$$

of (7.14), with
(7.29)
$$\gamma_\nu \stackrel{\Delta}{=} \frac{\sqrt{\rho_\nu^2 + 2(r-k\nu)} + \rho_\nu}{\sigma}, \quad \rho_\nu \stackrel{\Delta}{=} \frac{\beta + \nu - r}{\sigma} + \frac{\sigma}{2} \quad \text{and} \quad c_\nu \stackrel{\Delta}{=} \frac{q\gamma_\nu}{\gamma_\nu - 1}.$$

Clearly, γ_ν is the positive root of the equation $\frac{\sigma}{2}\xi^2 - \rho_\nu \xi - \frac{r-k\nu}{\sigma} = 0$, and we have the analogue $k < \gamma_\nu < (\frac{r-k\nu}{r-\beta-\nu}) \wedge \gamma$ of (7.8). Because $\nu \in (-\infty, 0]$, we can let $\nu \to -\infty$, and observe $\gamma_\nu \searrow k$, $c_\nu \to c$, $G_k^{(\nu)}(\cdot) \to G_k(\cdot)$, which leads to $h_{\text{up}}(K) \geq G_k(S(0))$.

5.6 Example: Prohibition of borrowing (continued). Here $K_+ = (-\infty, 1]$, $K_- = [1, \infty)$ and $\delta(x) + x = 0$ on $\tilde{K} = (-\infty, 0]$.

We claim that

$$h_{\text{up}}(K) = S(0). \tag{7.30}$$

The inequality $h_{\text{up}}(K) \leq S(0)$ is obvious; for the reverse inequality, we have

$$h_{\text{up}}(K) \geq \sup_{\tau \in \mathscr{S}_{0,\infty}} \mathbb{E}^\nu[e^{-(r-\nu)\tau}(S(\tau)-q)^+] =: G^{(\nu)}(S(0))$$

for every fixed $\nu \in \tilde{K}$. By analogy with (7.28) and (7.29), the optimal expected reward in this new stopping problem is given as

$$G^{(\nu)}(x) = \left\{ \begin{array}{ll} (c'_\nu - q)(\frac{x}{c'_\nu})^{\gamma'_\nu}; & 0 < x < c'_\nu \\ x - q; & c'_\nu \le x < \infty \end{array} \right\}$$

where now $\gamma'_\nu \stackrel{\Delta}{=} \frac{1}{\sigma}(\sqrt{\rho_\nu^2 + 2(r-\nu)} + \rho_\nu)$, $c'_\nu \stackrel{\Delta}{=} \frac{q\gamma'_\nu}{\gamma'_\nu - 1}$. Letting $\nu \downarrow -\infty$, we obtain now $\gamma'_\nu \searrow 1$, $c'_\nu \to \infty$, $G^{(\nu)}(x) \nearrow x$ and thus the desired inequality $h_{\text{up}}(K) \ge S(0)$.

We recall now the notation of (7.14)-(7.18) and claim that $u(0) = G(S(0))$ belongs to the set of (4.5); indeed, with $\check{\tau} \equiv \tau_b$ as in (7.16), $\check{C} \equiv 0$, and $\check{p}(\cdot) \equiv \hat{p}(\cdot) \in (1,\gamma] \subseteq [1,\infty) = K_-$ as in (7.15), we have $-X^{-u(0),\check{\pi},\check{C}}(\cdot) = \hat{X}_0(\cdot)$ on $[\![0,\check{\tau}]\!]$, as well as $(\check{\pi},\check{C}) \in \mathscr{A}_-(-u(0),\check{\tau})$ and (3.4) (since $X^{-u(0),\check{\pi},\check{C}}(\check{\tau}) + B(\check{\tau}) = -\hat{X}_0(\tau_b) + (S(\tau_b) - q)^+ = 0$, a.s). Thus, $h_{\text{low}}(K) \ge u(0)$, whereas the reverse inequality also holds, thanks to (4.6); we conclude that

$$h_{\text{low}}(K) = u(0) = G(S(0)). \tag{7.31}$$

8. A higher interest rate for borrowing

We have studied so far the hedging problem for American contingent claims in a financial market with the same interest rate for borrowing as for saving. However, the techniques developed in the previous sections can be adapted to a market $\mathscr{M}^*$ with interest rate $R(\cdot)$ for borrowing higher than the bond rate $r(\cdot)$ (saving rate).

We consider in this section an *unconstrained* market $\mathscr{M}^*$ with two different (bounded, $\mathbb{F}$-progressively measurable) interest rate processes $R(\cdot) \ge r(\cdot)$ for borrowing and saving, respectively. In this market $\mathscr{M}^*$, it is not reasonable to borrow money and to invest money in the bond, at the same time. Therefore, the relative amount borrowed at time t is equal to $(1 - \sum_{i=1}^d p_i(t))^-$. As shown in Cvitanić and Karatzas (1992), the wealth process $X(\cdot) = X^{x,\pi,C}(\cdot)$ corresponding to initial wealth x and a portfolio/consumption pair (π, C) as in Definition 2.3, satisfies now the analogue

$$\begin{aligned} dX(t) &= r(t)X(t)dt - dC(t) + X(t) \\ &\quad \times \left[p^*(t)\sigma(t)dW^{(0)}(t) - (R(t) - r(t))\left(1 - \sum_{i=1}^d p_i(t)\right)^- dt \right], \end{aligned}$$

of the wealth equation (2.9), whence

$$\begin{aligned} N(t) &\stackrel{\Delta}{=} \gamma_0(t)X(t) + \int_0^t \gamma_0(t)dC(t) \\ &\quad + \int_0^t \gamma_0(t)X(t)[R(t) - r(t)]\left(1 - \sum_{i=1}^d p_i(t)\right)^- dt, \quad 0 \le t \le T \end{aligned}$$

is a $\mathbb{P}^0$-local martingale.

The theory of Sect. 5 goes through with only very minor changes; namely, one sets $\delta(\nu(t)) = -\nu_1(t)$ for $\nu \in \mathscr{D}$, where $\mathscr{D}$ is now the class of $\mathbb{F}$-progressively measurable processes $\nu : [0,T] \times \Omega \to \mathscr{R}^d$ with

$$r(\cdot) - R(\cdot) \le \nu_1(\cdot) = \nu_2(\cdot) = \cdots = \nu_d(\cdot) \le 0$$

a.e. on $[0,T] \times \Omega$. With this notation, the statements of our main results, Theorems 5.12 and 5.13, continue to hold for the upper (h^*_{up})- and lower (h^*_{low})-hedging prices of the ACC $B(\cdot)$ in this market $\mathscr{M}^*$ with higher interest rate for borrowing:

$$h^*_{\text{up}} = \sup_{\nu \in \mathscr{D}} u_\nu(0), \quad h^*_{\text{low}} = \inf_{\nu \in \mathscr{D}} u_\nu(0). \tag{8.1}$$

Here

$$u_\nu(0) \overset{\Delta}{=} \sup_{\tau \in \mathscr{S}} \mathbb{E}^\nu[\gamma_\nu(\tau) B(\tau)] \tag{8.2}$$

is the unconstrained hedging price of Theorem 3.3 for the ACC $B(\cdot)$, in the market $\mathscr{M}_\nu$ with asset-prices governed by the equations

$$\left\{ \begin{array}{ll} dS_0^{(\nu)}(t) &= S_0^{(\nu)}(t)(r(t) - \nu_1(t))dt, \\ dS_i(t) &= S_i(t)\left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t) dW_j(t) \right], \quad i = 1, \ldots, d \end{array} \right\}$$

(that is, exactly as in (2.1), (2.2) but with interest-rate $r(\cdot) - \nu_1(\cdot)$ instead of $r(\cdot)$).

In particular, taking $\nu_1 \equiv (0, \ldots, 0)^*$ and $\nu_2 \equiv (r - R, \ldots, r - R)^*$ and setting

$$U(r) \overset{\Delta}{=} u_{\nu_1}(0) = \sup_{\tau \in \mathscr{S}} \mathbb{E}^{\nu_1}[\gamma_{\nu_1}(\tau) B(\tau)], \tag{8.3}$$

$$U(R) \overset{\Delta}{=} u_{\nu_2}(0) = \sup_{\tau \in \mathscr{S}} \mathbb{E}^{\nu_2}[\gamma_{\nu_2}(\tau) B(\tau)] \tag{8.4}$$

we obtain

$$h^*_{\text{low}} \le U(r) \wedge U(R) \le U(r) \vee U(R) \le h^*_{\text{up}}. \tag{8.5}$$

Clearly, $U(r)$ and $U(R)$ are the arbitrage-free prices of (3.8), with unconstrained portfolios, corresponding to interest rate processes $r(\cdot)$ and $R(\cdot)$, respectively.

In the special case of an *American call-option* $B(t) = (S_1(t) - q)^+$ and constant $R(\cdot) \equiv R > r \equiv r(\cdot)$, we know from Examples 1.4.7, 1.3.2 in Karatzas (1997) that the optimal unconstrained hedging portfolio $\hat{p}(\cdot)$ for the buyer always borrows (at the interest rate R, since $\hat{p}_1(\cdot) > 1$), whereas the optimal hedging portfolio $\check{\pi}(\cdot) = -\hat{\pi}(\cdot)$ for the seller always saves (at the interest rate r). Consequently, $U(R) \ge h^*_{\text{up}}, \quad U(r) \le h^*_{\text{low}}$, and in conjunction with (8.4):

$$h^*_{\text{up}} = U(R), \quad h^*_{\text{low}} = U(r). \tag{8.6}$$

In other words, the upper (respectively, lower) hedging price h^*_{up} (respectively, h^*_{low}) of the American call-option $B(\cdot) = (S(\cdot) - q)^+$ in the market $\mathscr{M}^*$, is given by the Black-Scholes formula (e.g. p. 19 in Karatzas (1997)), evaluated at the higher interest-rate R (respectively, at the lower interest-rate r).

Appendix A: Proof of Theorem 5.12

We shall assume throughout this section that the quantity V *of (5.9) is finite*, and show that

$$(A.1) \qquad \text{there exists a pair } (\hat{\pi}, \hat{C}) \in \mathscr{A}_+(V) \text{ which satisfies (5.24), a.s.}$$

This will imply $h_{up}(K) \leq V$, and complete the proof of Theorem 5.12.

Let us start by introducing the family of random variables

$$(A.2) \quad \tilde{X}(\tau) \stackrel{\Delta}{=} \text{ess sup}_{\nu \in \mathscr{D}} \text{ess sup}_{\rho \in \mathscr{S}_{\tau,T}} \frac{1}{\gamma_\nu(\tau)} E^\nu[\gamma_\nu(\rho) B(\rho) | \mathscr{F}(\tau)], \quad \tau \in \mathscr{S}$$

with $\tilde{X}(0) = V$, $\tilde{X}(T) = B(T)$ a.s.; here $\mathscr{S}_{\tau,\rho}$ denotes the class of stopping times ξ with $\tau \leq \xi \leq \rho$ a.s., for any two stopping times τ, ρ such that $\mathbb{P}[\tau \leq \rho] = 1$. For every fixed $t \in [0, T]$ we have $\mathbb{P}[\tilde{X}(t) = \overline{X}(t)] = 1$ from (5.23) and (A.2). Note also that $\tilde{X}(\cdot)$ is the value of a *double* (optimal stopping over $\rho \in \mathscr{S}$/stochastic control over $\nu \in \mathscr{D}$) *stochastic maximization problem*. For notational convenience, we shall introduce also the random variable

$$(A.3) \quad I(\tau|\rho, \nu) \stackrel{\Delta}{=} \frac{1}{\gamma_\nu(\tau)} \mathbb{E}^\nu[\gamma_\nu(\rho) B(\rho) | \mathscr{F}(\tau)] = \mathbb{E}[Z_\nu(\tau, \rho)\gamma_\nu(\tau, \rho) B(\rho) | \mathscr{F}(\tau)]$$

for every $\tau \in \mathscr{S}$, $\rho \in \mathscr{S}_{\tau,T}$, $\nu \in \mathscr{D}$, where $Z_\nu(\tau, \rho) \stackrel{\Delta}{=} Z_\nu(\rho)/Z_\nu(\tau)$, $\gamma_\nu(\tau, \rho) \stackrel{\Delta}{=} \gamma_\nu(\rho)/\gamma_\nu(\tau)$. Clearly, the random variable of (A.3) depends only on the restriction of the process $\nu(\cdot) \in \mathscr{D}$ to the stochastic interval

$$(A.4) \qquad [\![\tau, \rho]\!] \stackrel{\Delta}{=} \{(t, w) \in [0, T]/\tau(w) \leq t \leq \rho(w)\}.$$

We shall denote by $\mathscr{D}_{\tau,\rho}$ the restriction of $\mathscr{D}$ to this stochastic interval.

We know from a fundamental property of the essential supremum (e.g. Neveu (1975), p.121) that

$$(A.5) \qquad \tilde{X}(\tau) = \lim_{k \to \infty} I(\tau|\rho_k, \nu_k), \text{ a.s.}$$

for some sequence $\{(\rho_k, \nu_k)\}_{k \in \mathbb{N}}$, where $\rho_k \in \mathscr{S}_{\tau,T}$, $\nu_k \in \mathscr{D}_{\tau,\rho_k}$, $\forall\, k \in \mathbb{N}$.

A.1 Lemma: *For every fixed $\nu(\cdot) \in \mathscr{D}$, and stopping times $\tau \leq \rho$ in $\mathscr{S}$, we have*

$$(A.6) \qquad \gamma_\nu(\tau)\tilde{X}(\tau) \geq \mathbb{E}^\nu[\gamma_\nu(\rho)\tilde{X}(\rho) | \mathscr{F}(\tau)], \text{ a.s.}$$

Proof: Let $\mathscr{N}_{\tau,\rho}$ be the class of processes $\mu(\cdot)$ in $\mathscr{D}$, which agree with $\nu(\cdot)$ on the stochastic interval $[\![\tau, \rho]\!]$ of (A.4). For every process $\mu(\cdot)$ in $\mathscr{N}_{\tau,\rho}$, and $\xi \in \mathscr{S}_{\rho,T}$, we have then

$$I(\tau|\xi, \mu) = \mathbb{E}[Z_\mu(\tau, \xi)\gamma_\mu(\tau, \xi) B(\xi) | \mathscr{F}(\tau)]$$

$$= \mathbb{E}[Z_\nu(\tau,\rho)\gamma_\nu(\tau,\rho).\,\mathbb{E}\{Z_\mu(\rho,\xi)\gamma_\mu(\rho,\xi)B(\xi)|\mathscr{F}(\rho)\}|\mathscr{F}(\tau)]$$
$$= \mathbb{E}[Z_\nu(\tau,\rho)\gamma_\nu(\tau,\rho)\,I(\rho|\xi,\mu)|\mathscr{F}(\tau)]$$
$$(A.7)\qquad = \frac{1}{\gamma_\nu(\tau)}\mathbb{E}^\nu[\gamma_\nu(\rho)I(\rho|\xi,\mu)|\mathscr{F}(\tau)]$$

almost surely. Now from (A.5), there exists a sequence $\{(\xi_k,\mu_k)\}_{k\in\mathbb{N}}$ with $\xi_k \in \mathscr{S}_{\rho,T}$ and $\mu_k \in \mathscr{D}_{\rho,\xi_k}$ ($\forall k \in \mathbb{N}$), such that $\tilde{X}(\rho) = \lim_k I(\rho|\xi_k,\mu_k)$ a.s. Thus, without loss of generality, we may take $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \mathscr{N}_{\tau,\rho}$ and obtain with the help of Fatou's lemma

$$\begin{aligned}\tilde{X}(\tau) &\geq \text{ess sup}_{\mu\in\mathscr{N}_{\tau,\rho}}\text{ess sup}_{\xi\in\mathscr{S}_{\rho,T}} I(\tau|\xi,\mu)\\ &\geq \underline{\lim}_{k\to\infty} I(\tau|\xi_k,\mu_k)\\ &\geq \frac{1}{\gamma_\nu(\tau)}\mathbb{E}^\nu[\gamma_\nu(\rho).\,\underline{\lim}_{k\to\infty} I(\rho|\xi_k,\mu_k)|\mathscr{F}(\tau)]\\ &= \frac{1}{\gamma_\nu(\tau)}\mathbb{E}^\nu[\gamma_\nu(\rho)\tilde{X}(\rho)|\mathscr{F}(\tau)],\ \text{a.s.}\end{aligned}$$

A.2 Lemma: *The $\mathbb{F}$-adapted process $\overline{X}(\cdot)$ of (5.23) can be considered in its RCLL modification, and $\gamma_\nu(\cdot)\overline{X}(\cdot)$ is a $\mathbb{P}^\nu$-supermartingale for every $\nu \in \mathscr{D}$. With $\overline{X}(\cdot)$ in this RCLL modification we have $\tilde{X}(\tau) = \overline{X}(\tau)$ a.s, for every $\tau \in \mathscr{S}$.*

Proof: The supermartingale property follows directly from (A.6) with deterministic $\tau = t$, $\rho = s$ $(0 \leq t \leq s \leq T)$, since $\tilde{X}(t) = \overline{X}(t)$ and $\tilde{X}(s) = \overline{X}(s)$ a.s. The RCLL regularity is proved as in El Karoui and Quenez (1995); finally, the last assertion follows as in Karatzas and Shreve (1998), Appendix D or Remark 5.6.7.

Proof of (A.1) (adapted from Cvitanić and Karatzas (1993)): From the Doob-Meyer decomposition, and the martingale representation property of the Brownian filtration $\mathbb{F}$, we can represent the supermartingale $\gamma_\nu(\cdot)\overline{X}(\cdot)$ of Lemma A.2 as

$$(A.8)\qquad \gamma_\nu(t)\overline{X}(t) = V + M_\nu(t) - A_\nu(t) = V + \int_0^t \psi_\nu^*(s)dW^{(\nu)}(s) - A_\nu(t), \quad 0 \leq t \leq T$$

for every $\nu \in \mathscr{D}$. Here $\psi_\nu : [0,T]\times\Omega \to \mathscr{R}^d$ is $\mathbb{F}$-progressively measurable with $\int_0^T \|\psi_\nu(t)\|^2 dt < \infty$ a.s., and $A_\nu : [0,T]\times\Omega \to [0,\infty)$ is $\mathbb{F}$-adapted, natural increasing, with right-continuous paths and $A_\nu(0) = 0$. We interpret (A.8) as a *simultaneous Doob-Meyer decomposition*, valid for every $\nu \in \mathscr{D}$.

Consider now an arbitrary $\mu \in \mathscr{D}$ and observe, thanks to $\gamma_\mu(t)/\gamma_\nu(t) = \exp\left(\int_0^t(\delta(\nu(s)) - \delta(\mu(s)))ds\right)$ and $dW^{(\nu)}(t) = dW^{(\mu)}(t) + \sigma^{-1}(t)(\nu(t) - \mu(t))dt$ (from (5.14), (5.17) respectively), that (A.8) gives

$$\gamma_\mu(t)\overline{X}(t) = V + \int_0^t \frac{\gamma_\mu(s)}{\gamma_\nu(s)}\psi_\nu^*(s)dW^{(\mu)}(s) - \int_{(0,t]}\frac{\gamma_\mu(s)}{\gamma_\nu(s)}dA_\nu(s)$$
$$(A.9)\qquad - \int_0^t \frac{\gamma_\mu(s)}{\gamma_\nu(s)}\left[\psi_\nu^*(s)\sigma^{-1}(s)(\mu(s) - \nu(s)) + \gamma_\nu(s)\overline{X}(s)(\delta(\mu(s)) - \delta(\nu(s)))\right]ds.$$

Comparing this expression with the analogue of (A.8)

$$\gamma_\nu(t)\overline{X}(t) = V + \int_0^t \psi_\mu^*(s)dW^{(\mu)}(s) - A_\mu(t), \quad 0 \le t \le T$$

for $\mu(\cdot)$, we obtain that *the processes*

$$(A.10) \qquad \frac{\psi_\mu(t)}{\gamma_\mu(t)} = \frac{\psi_\nu(t)}{\gamma_\nu(t)} =: h(t), \quad 0 \le t \le T$$

$$\int_{(0,t]} \frac{dA_\mu(s)}{\gamma_\mu(s)} - \int_0^t [\overline{X}(s)\delta(\mu(s)) + h^*(s)\sigma^{-1}(s)\mu(s)]ds$$
$$= \int_{(0,t]} \frac{dA_\nu(s)}{\gamma_\nu(s)} - \int_0^t [\overline{X}(s)\delta(\nu(s))$$
$$(A.11) \qquad + h^*(s)\sigma^{-1}(s)\nu(s)]ds =: \hat{C}(t), \quad 0 \le t \le T$$

do not depend on $\nu \in \mathscr{D}$. In particular, $\hat{C}(t) \equiv \int_{(0,t]} \frac{dA_0(s)}{\gamma_0(s)}$, $0 \le t \le T$ has increasing, right-continuous paths. We also have

$$(A.12) \qquad \int_0^T 1_{\{\bar{X}(t)=0\}} \|h(t)\|^2 dt = \int_0^T \gamma_\nu^{-2}(t) 1_{\{\bar{X}(t)=0\}} d <M_\nu> (t) = 0, \text{ a.s.}$$

from equations (12.1), (12.3), p. 365 in Meyer (1976); see also Karatzas and Shreve (1991), p.225, Exercise 7.10. It develops then that the portfolio process $\hat{\pi} : [0,T] \times \Omega \to \mathscr{R}^d$ and the portfolio-proportion process $\hat{p} : [0,T] \times \Omega \to \mathscr{R}^d$

$$(A.13) \quad \hat{\pi}(t) \stackrel{\Delta}{=} (\sigma^*(t))^{-1} h(t), \quad \hat{p}_i(t) \stackrel{\Delta}{=} \frac{\hat{\pi}_i(t)}{X(t)} 1_{\{\bar{X}(t)>0\}}, \; i = 1, \ldots, d, \quad 0 \le t \le T$$

are $\mathbb{F}$-progressively measurable and satisfy $\int_0^T \|\hat{\pi}^*(t)\|^2 dt < \infty$ and

$$(A.14) \qquad h^*(t) = \psi_\nu^*(t)/\gamma_\nu(t) = \overline{X}(t)\hat{p}^*(t)\sigma(t), \quad 0 \le t \le T$$

almost surely. The same arguments as on p.664 of Cvitanić and Karatzas (1993), based on the fact that $A_\nu(t) = \int_{(0,t]} \gamma_\nu(s) d\hat{C}(s) + \int_0^t \overline{X}(s)[\delta(\nu(s)) + \nu^*(s)\hat{p}(s)]ds$ of (A.11), (A.13) is an increasing process, yield $\delta(\nu(\cdot)) + \nu^*(\cdot)\hat{p}(\cdot) \ge 0$, a.e. for every $\nu \in \mathscr{D}$. On the other hand, the proof on pp.782-783 of Cvitanić and Karatzas (1992), along with the continuity condition (5.7), the fact that K_+ is closed, and Theorem 13.1, p.112 in Rockafellar (1970), show that

$$(A.15) \qquad \hat{p}(t) \in K_+, \quad \forall\, 0 \le t \le T$$

holds a.s. Finally, substitution of (A.14) back into (A.11), (A.8) leads to

$$\gamma_\nu(t)\overline{X}(t) + \int_{(0,t]} \gamma_\nu(s) d\hat{C}(s) + \int_0^t \gamma_\nu(s)\overline{X}(s)(\delta(\nu(s)) + \nu^*(s)\hat{p}(s))ds$$
$$(A.16) \qquad = V + \int_0^t \gamma_\nu(s)\overline{X}(s)\hat{p}^*(s)\sigma(s)dW^{(\nu)}(s), \quad 0 \le t \le T$$

for every $\nu \in \mathscr{D}$.

A comparison of (A.16) with (5.21) shows that $X^{V,\hat{\pi},\hat{C}}(\cdot) \equiv \overline{X}(\cdot)$; since $\overline{X}(\cdot) \geq B(\cdot) \geq 0$ and (A.15) hold, we conclude that the portfolio/consumption process pair $(\hat{\pi}, \hat{C})$ in (A.11), (A.13) belongs to the class $\mathscr{A}_+(V)$ of (4.2). ◻

Appendix B: Proof of Theorem 5.13

Let us start by introducing the family of random variables

$$\underline{X}(\tau) \stackrel{\Delta}{=} \text{ess inf}_{\nu\in\mathscr{D}} \hat{X}_\nu(\tau),$$

$$(B.1) \qquad \hat{X}_\nu(\tau) = \text{ess sup}_{\rho\in\mathscr{S}_{\tau,T}} \mathbb{E}^\nu\left[\frac{\gamma_\nu(\rho)}{\gamma_\nu(\tau)} B(\rho)\middle|\mathscr{F}(\tau)\right], \quad \tau \in \mathscr{S}$$

with $\underline{X}(0) = v,\ \underline{X}(T) = B(T)$ a.s. Note that $\underline{X}(\cdot)$ is the upper-value of a *stochastic game*, in which one player (the "maximizer") selects the stopping time ρ, and the other player (the "minimizer") gets to choose the stochastic process $\nu \in \mathscr{D}$. Arguing as in Remark 3.2, and using (3.1), we obtain for $1 < p < 1+\epsilon$:

$$(B.2) \quad \mathbb{E}^\nu\left[\sup_{\tau\in\mathscr{S}} (\gamma_\nu(\tau)\underline{X}(\tau))^p\right] \leq \mathbb{E}^\nu\left[\sup_{\tau\in\mathscr{S}} (\gamma_\nu(\tau)\hat{X}_\nu(\tau))^p\right] < \infty, \quad \forall\, \nu \in \mathscr{D}.$$

Prooof of (B.2): For every $\nu \in \mathscr{D},\ \tau \in \mathscr{S}$ and with

$$Y_\nu(t) \stackrel{\Delta}{=} \mathbb{E}^\nu[Y|\mathscr{F}(t)],\ 0 \leq t \leq T, \quad Y \stackrel{\Delta}{=} \sup_{0\leq t\leq T} (\gamma_0(t)B(t)),$$

we have $\mathbb{E}(Y^{1+\epsilon}) < \infty$ from (3.1) and

$$\gamma_\nu(\tau)\hat{X}_\nu(\tau) \leq \mathbb{E}^\nu\left[\sup_{0\leq t\leq T} (\gamma_\nu(t)B(t))\middle|\mathscr{F}(\tau)\right] \leq c\cdot \mathbb{E}^\nu[Y|\mathscr{F}(\tau)] = cY_\nu(\tau), \text{ a.s.}$$

Thus, from the Doob maximal martingale inequality and the Hölder inequality,

$$\mathbb{E}^\nu\left[\sup_{\tau\in\mathscr{S}} (\gamma_\nu(\tau)\hat{X}_\nu(\tau))^p\right] \leq c\cdot \mathbb{E}^\nu\left[\sup_{0\leq t\leq T} (Y_\nu(t))^p\right] \leq c\cdot \mathbb{E}^\nu(Y^p)$$
$$= c\cdot \mathbb{E}[Z_\nu(T)Y^p] \leq c\cdot (\mathbb{E}(Y^{1+\epsilon}))^{1/r}(\mathbb{E}(Z_\nu(T))^s)^{1/s} \leq c < \infty$$

with $r = \frac{1+\epsilon}{p},\ \frac{1}{r} + \frac{1}{s} = 1$. In the above, $c = c_\nu$ is a real constant which depends on $\nu \in \mathscr{D}$ and is allowed to vary from line to line.

B.1 Lemma: *Suppose $V < \infty$. For any stopping times $\tau \leq \rho \leq \rho_0$, with ρ_0 as in (5.26), we have*

$$(B.3) \qquad \gamma_\nu(\tau)\underline{X}(\tau) \leq \mathbb{E}^\nu[\gamma_\nu(\rho)\underline{X}(\rho)|\mathscr{F}(\tau)], \text{ a.s.} \quad (\forall\, \nu \in \mathscr{D})$$

$$(B.4) \qquad \underline{X}(\tau) = \text{ess inf}_{\nu\in\mathscr{D}} \frac{1}{\gamma_\nu(\tau)} \mathbb{E}[\gamma_\nu(\rho_0)B(\rho_0)|\mathscr{F}(\tau)], \text{ a.s.}$$

In particular, for every $\nu \in \mathscr{D}$, the $\mathbb{F}$-adapted process

$(B.5)\qquad Q_\nu(t) \stackrel{\Delta}{=} \gamma_\nu(t\wedge\rho_0)\underline{X}(t\wedge\rho_0),\quad 0\le t\le T$ *is a* $\mathbb{P}^\nu$*-submartingale*

and can be considered, along with $\underline{X}(\cdot\wedge\rho_0)$*, in its RCLL modification.*

Proof: Fix a process $\nu(\cdot)\in\mathscr{D}$ and consider a sequence of processes $\{\nu_k(\cdot)\}_{k\in\mathbb{N}}\subseteq \mathscr{D}_{\rho,T}\cap\mathscr{N}_{\tau,\rho}$, which agree with $\nu(\cdot)$ on $[\![\tau,\rho]\!]$ (notation of (A.4) and of Proof, Lemma A.1) and are such that $\underline{X}(\rho)=\lim_k \hat{X}_{\nu_k}(\rho)$, a.s.

The sequence $\{\gamma_{\nu_k}(\tau,\rho)\hat{X}_{\nu_k}(\rho)\}_{k\in\mathbb{N}}$ is dominated by the random variable $\gamma_\nu(\tau,\rho)\overline{X}(\rho)$ which is $\mathbb{P}^\nu$-integrable since, from (A.6),

$$\mathbb{E}^\nu[\gamma_\nu(\tau,\rho)\overline{X}(\rho)]\le c\cdot\mathbb{E}^\nu[\gamma_\nu(\rho)\overline{X}(\rho)]\le c\cdot\gamma_\nu(0)\overline{X}(0)=c\cdot V<\infty.$$

Therefore, the Dominated Convergence Theorem gives

$$\begin{aligned}
\frac{1}{\gamma_\nu(\tau)}\mathbb{E}^\nu[\gamma_\nu(\rho)\underline{X}(\rho)|\mathscr{F}(\tau)] &= \mathbb{E}^\nu\left[\gamma_\nu(\tau,\rho)\lim_k \hat{X}_{\nu_k}(\rho)\middle|\mathscr{F}(\tau)\right]\\
&= \mathbb{E}^\nu\left[\lim_k(\gamma_{\nu_k}(\tau,\rho)\hat{X}_{\nu_k}(\rho))\middle|\mathscr{F}(\tau)\right]=\lim_k\mathbb{E}^\nu[\gamma_{\nu_k}(\tau,\rho)\hat{X}_{\nu_k}(\rho)|\mathscr{F}(\tau)]\\
&= \lim_k\frac{1}{\gamma_{\nu_k}(\tau)}\mathbb{E}^{\nu_k}[\gamma_{\nu_k}(\rho)\hat{X}_{\nu_k}(\rho)|\mathscr{F}(\tau)]\\
&\ge \text{ess inf}_{\mu\in\mathscr{D}}\frac{1}{\gamma_\mu(\tau)}\mathbb{E}^\mu[\gamma_\mu(\rho)\hat{X}_\mu(\rho)|\mathscr{F}(\tau)]\\
&= \text{ess inf}_{\mu\in\mathscr{D}}\hat{X}_\mu(\tau)=\underline{X}(\tau),\ \text{a.s.}
\end{aligned}$$

The next-to-last equality follows from the fact that for every $\mu\in\mathscr{D}$,

$$(B.5)'\qquad \gamma_\mu(\cdot\wedge\rho_0)\hat{X}_\mu(\cdot\wedge\rho_0)\ \text{is a}\ \mathbb{P}^\mu\text{-martingale}$$

(recall Property (ii) in Theorem 1.4.4 of Karatzas (1997), the notation of (5.30)), and $\rho_0\le\check{\rho}_0(\mu)$ a.s.). This proves (B.3).

To prove (B.4), observe that the a.s. inequality

$$\underline{X}(\tau)\ge\text{ess inf}_{\nu\in\mathscr{D}}\mathbb{E}\left[\frac{\gamma_\nu(\rho_0)}{\gamma_\nu(\tau)}B(\rho_0)\middle|\mathscr{F}(\tau)\right]$$

follows directly from (B.1); to prove the reverse inequality, one has to show

$$\underline{X}(\tau)\le\frac{1}{\gamma_\nu(\tau)}\mathbb{E}^\nu[\gamma_\nu(\rho_0)B(\rho_0)|\mathscr{F}(\tau)]\ \text{a.s.}\quad(\forall\,\nu\in\mathscr{D}),$$

but this follows directly from (B.3) and $B(\rho_0)=\underline{X}(\rho_0)$ a.s. The almost sure representation

$$(5.33)\qquad \underline{X}(t)=\text{ess inf}_{\nu\in\mathscr{D}}\frac{1}{\gamma_\nu(t)}\mathbb{E}[\gamma_\nu(\rho_t)B(\rho_t)|\mathscr{F}(t)],\quad 0\le t\le T$$

is then proved in exactly the same way as (B.4). And (B.5) is a direct consequence of (B.3).

Now arguing exactly as in Lemma A.2, we conclude that the processes $Q_\nu(t+)=\gamma_\nu(t)\underline{X}(t+)$, $0\le t<T$ and $Q_\nu(t-)=\gamma_\nu(t)\underline{X}(t-)$, $0<t\le T$

are well-defined, and are $\mathbb{P}^\nu$-submartingales for every $\nu \in \mathscr{D}$. Therefore, from (B.5), (B.2) and Fatou's lemma we have, on the event $\{t < \rho_0\}$:

$$\begin{aligned}
\underline{X}(t) &\le \frac{1}{\gamma_0(t)} \overline{\lim}_n \mathbb{E}^0 \left[\gamma_0\left(t+\frac{1}{n}\right) \underline{X}\left(t+\frac{1}{n}\right) \Big| \mathscr{F}(t) \right] \\
&= \frac{1}{\gamma_0(t)} \overline{\lim}_n \mathbb{E}^0 \left[Q_0\left(t+\frac{1}{n}\right) \Big| \mathscr{F}(t) \right] \\
&\le \frac{1}{\gamma_0(t)} \mathbb{E}^0 \left[\overline{\lim}_n Q_0\left(t+\frac{1}{n}\right) \Big| \mathscr{F}(t) \right] = \frac{1}{\gamma_0(t)} Q_0(t+) = \underline{X}(t+),
\end{aligned}$$

a.s. On this same event $\{t < \rho_0\}$ we have also $\rho_t = \rho_0$ and

$$Q_\nu(t+) \le \mathbb{E}^\nu[Q_\nu(\rho_0+)|\mathscr{F}(t)] = \mathbb{E}^\nu[\gamma_\nu(\rho_0)B(\rho_0)|\mathscr{F}(t)] = \mathbb{E}^\nu[\gamma_\nu(\rho_t)B(\rho_t)|\mathscr{F}(t)]$$

almost surely, whence $\underline{X}(t+) \le \text{ess sup}_{\nu\in\mathscr{D}} \frac{1}{\gamma_\nu(t)} \mathbb{E}^\nu[\gamma_\nu(\rho_t)B(\rho_t)|\mathscr{F}(t)] = \underline{X}(t)$, a.s. from (5.33). This shows that $\underline{X}(\cdot \wedge \rho_0)$ can be considered in its RCLL modification.

□

Proof of Theorem 5.13 (continued): In order to complete the proof of Theorem 5.13, it remains to show the inequality

$$(B.6) \qquad v \le h_{\text{low}}(K), \text{ whenever } v > 0 \text{ and } V < \infty.$$

Thus, let us assume from now on that both $v > 0$, $V < \infty$ hold, and observe that for each $\nu \in \mathscr{D}$, the $\mathbb{P}^\nu$-submartingale $Q_\nu(\cdot)$ of (B.5) has RCLL paths and is of class $\mathscr{D}[0,T]$ under $\mathbb{P}^\nu$ (recall Lemma B.1 and (B.2)). From the Doob-Meyer decomposition (Karatzas and Shreve (1991), Sect. 3.5) we can write this process in the form

$$(B.7) \qquad Q_\nu(t) = v + M_\nu(t) + A_\nu(t), \quad 0 \le t \le T$$

where $M_\nu(t) = \int_0^t \psi_\nu^*(s) dW^{(\nu)}(s)$, $0 \le t \le T$ is a $\mathbb{P}^\nu$-martingale, $\psi_\nu : [0,T] \times \Omega \to \mathscr{R}^d$ is an $\mathbb{F}$-progressively measurable process with $\int_0^T \|\psi_\nu(t)\|^2 dt < \infty$ a.s., and $A_\nu(\cdot)$ is an $\mathbb{F}$-adapted natural increasing process with right-continuous paths and $A_\nu(0) = 0$, $\mathbb{E}^\nu A_\nu(T) < \infty$. Again, (B.7) is another *simultaneous Doob-Meyer decomposition*, valid for all $\nu \in \mathscr{D}$. Clearly, we may take

$$(B.8) \qquad \psi_\nu(\cdot) \equiv 0 \text{ a.e. on } [\![\rho_0, T]\!], \text{ and } A_\nu(\rho_0) = A_\nu(T) \text{ a.s.}$$

Proceeding exactly as in the proof of (A.1) in Appendix A, we see here again that

$$(B.9) \qquad h(t) \stackrel{\Delta}{=} \frac{\psi_\nu(t)}{\gamma_\nu(t)}, \quad 0 \le t \le T$$

$$(B.10) \quad \check{C}(t) \stackrel{\Delta}{=} \int_{(0,t]} \frac{dA_\nu(s)}{\gamma_\nu(s)} + \int_0^t (\underline{X}(s)\delta(\nu(s)) + h^*(s)\sigma^{-1}(s)\nu(s))ds, \quad 0 \le t \le T$$

do not depend on $\nu \in \mathscr{D}$ by analogy with (A.10) and (A.11); in particular, $\check{C}(\cdot) = \int_{(0,\cdot]} \frac{dA_0(s)}{\gamma_0(s)}$ has increasing and right-continuous paths. We also have the analogue

$$\int_0^T 1_{\{\underline{X}(t)=0\}} \|h(t)\|^2 dt = 0, \text{ a.s.} \tag{B.11}$$

of (A.12). Thus, if we define an $\mathbb{F}$-progressively measurable process $\check{\pi} : [0,T] \times \Omega \to \mathscr{R}^d$ and $\check{p} : [0,T] \times \Omega \to \mathscr{R}^d$ via

(B.12)

$$\check{\pi}(t) \overset{\Delta}{=} (\sigma^*(t))^{-1} h(t), \quad \check{p}_i(t) \overset{\Delta}{=} \frac{1}{\underline{X}(t)} \check{\pi}_i(t) 1_{\{\underline{X}(t)>0\}}, \quad i = 1, \ldots, d, \quad 0 \le t \le T \ ,$$

we have $\int_0^T \|\check{\pi}^*(t)\|^2 dt < \infty$ a.s., $h(\cdot) \equiv 0$ a.e. on $[\![\rho_0, T]\!]$, $\check{C}(\rho_0) = \check{C}(T)$ a.s.,

$$h^*(\cdot) = \frac{\psi_\nu^*(\cdot)}{\gamma_\nu(\cdot)} = \underline{X}(\cdot)(\check{p}(\cdot))^* \sigma(\cdot) \qquad \text{a.e. on } [0,T] \tag{B.13}$$

as well as

(B.14)

$$A_\nu(t) = \int_{(0,t]} \gamma_\nu(s) d\check{C}(s) - \int_0^t \gamma_\nu(s) \underline{X}(s)[\delta(\nu(s)) + (\check{p}(s))^* \nu(s)] ds, \quad 0 \le t \le T$$

a.e. The same arguments as on p.664 of Cvitanić and Karatzas (1993) now yield $\delta(\nu(\cdot)) + (\check{p}(\cdot))^* \nu(\cdot) \le 0$ a.e., for every $\nu \in \mathscr{D}$; and the proof on pp. 782-783 of Cvitanić and Karatzas (1992), along with (5.5), (5.7), the fact that K_- is closed, and Theorem 13.1 on p.112 of Rockafellar (1970), show that

$$\check{p}(\cdot) \in K_-, \text{ a.e.} \tag{B.15}$$

Now let us substitute (B.12)-(B.14) back into (B.7) and set $\nu(\cdot) \equiv 0$, to obtain

$$\begin{aligned} \gamma_0(t \wedge \rho_0)(-\underline{X}(t \wedge \rho_0)) &= (-\upsilon) - \int_{(0,t]} \gamma_0(s) d\check{C}(s) \\ &\quad + \int_0^t \gamma_0(s)(-\underline{X}(s))(\check{p}(s))^* dW^{(0)}(s), \quad 0 \le t \le T. \end{aligned} \tag{B.16}$$

In other words,

$$-\underline{X}^{-\upsilon,\check{\pi},\check{C}}(t) \equiv \left\{ \begin{array}{ll} \underline{X}(t) & ; 0 \le t < \rho_0 \\ \underline{X}(\rho_0) \cdot \frac{\gamma_0(\rho_0)}{\gamma_0(t)} & ; \rho_0 \le t \le T \end{array} \right\} \tag{B.17}$$

almost surely; and we conclude from (B.15) that the pair $(\check{\pi}, \check{C})$ belongs to the class $\mathscr{A}_-(-\upsilon)$ of (4.3). On the other hand, $-X^{-\upsilon,\check{\pi},\check{C}}(\rho_0) = \underline{X}(\rho_0) = B(\rho_0)$, a.s., so that the condition (3.4) holds with $\check{\tau} = \rho_0$. It develops that υ belongs to the set of (4.5), and thus (B.6) follows. □

Proof of Proposition 5.14: We have already proved (5.33), and this leads directly to (5.35) and (5.34). For (5.32), observe

$$(B.18)\quad \hat{X}_\nu(t) = \mathbb{E}^\nu\left[\frac{\gamma_\nu(\rho_t)}{\gamma_\nu(t)}\hat{X}_\nu(\rho_t)\middle|\mathscr{F}(t)\right] \le \mathbb{E}^\nu\left[\frac{\gamma_\nu(\rho_t)}{\gamma_\nu(t)}\overline{X}(\rho_t)\middle|\mathscr{F}(t)\right] \le \overline{X}(t),\ \text{a.s.}$$

for every $\nu \in \mathscr{D}$, thanks to (B.5)′ and Lemma A.1; now (5.32) follows by taking essential suprema in (B.18) over $\nu \in \mathscr{D}$, in conjunction with Lemma A.2, and leads to (5.36). □

Appendix C: Proof of Theorems 6.1 and 6.2

In this section we provide proofs of Theorems 6.1 and 6.2, following Broadie et al. (1996); see also Karatzas and Shreve (1998), Sects. 5.7 and 5.10. In order to simplify notation a little, we shall only deal with the one-dimensional case $d = 1$ (the general case requires only more complicated notation), and we shall write

$$S(t) = xe^{-\int_0^t \nu(s)ds}\Gamma_\nu(t), \qquad \Gamma_\nu(t) \stackrel{\Delta}{=} \exp\left[\sigma W_\nu(t) + \left(r - \sigma^2/2\right)t\right];\ \ 0 \le t \le T$$

for every process $\nu(\cdot) \in \mathscr{D}$, and with $x = S(0) > 0$ the initial stock price.

C.1 Lemma: *For every $\nu(\cdot) \in \mathscr{D}$, $\tau \in \mathscr{S}$, we have, almost surely,*

$$(C.1)\qquad \delta\left(\int_0^\tau \nu(s)\,ds\right) = \int_0^\tau \delta(\nu(s))\,ds.$$

Proof: From the definition (5.2) of the support function $\delta(\cdot)$, we have for every $\nu(\cdot) \in \mathscr{D}$, $\tau \in \mathscr{S}$,

$$\begin{aligned}\delta\left(\int_0^\tau \nu(s)\,ds\right) &= \sup_{p\in K_+}\left(-\int_0^\tau p\nu(s)\,ds\right)\\ &\le \int_0^\tau \sup_{p\in K_+}(-p\nu(s))\,ds \le \int_0^\tau \delta(\nu(s))\,ds\end{aligned}$$

almost surely; on the other hand, from (5.5) we obtain

$$\begin{aligned}\delta\left(\int_0^\tau \nu(s)\,ds\right) &= \inf_{p\in K_-}\left(\int_0^\tau -p\nu(s)\,ds\right)\\ &\ge \int_0^\tau \inf_{p\in K_-}(-p\nu(s))\,ds = \int_0^\tau \delta(\nu(s))\,ds. \qquad \square\end{aligned}$$

Another crucial observation here is that

$$(C.2)\quad \left\{\begin{array}{c}\text{the law of } \Gamma_\nu(t),\ \ 0 \le t \le T \text{ under } \mathbb{P}^\nu,\ \ \text{is the same as the law}\\ \text{of } \Gamma_0(t), 0 \le t \le T \text{ under } \mathbb{P}^0,\ \ \text{for any } \nu(\cdot) \in \mathscr{D}.\end{array}\right\}$$

Proof of Theorem 6.1: From (C.1) and (C.2) we obtain

$$\mathbb{E}^{\nu}\left[\exp\left\{-r\tau-\int_0^{\tau}\delta(\nu(s))\,ds\right\}\varphi(S(\tau))\right]$$
$$=\mathbb{E}^{\nu}\left[\exp\left\{-r\tau-\delta\left(\int_0^{\tau}\nu(s)\,ds\right)\right\}\varphi\left(xe^{-\int_0^{\tau}\nu(s)ds}\Gamma_{\nu}(\tau)\right)\right]$$
$$\leq\mathbb{E}^{\nu}\left[e^{-r\tau}\bar{\varphi}(x\Gamma_{\nu}(\tau))\right]$$
$$(C.3)\qquad \leq\sup_{\tau\in\mathscr{S}}\mathbb{E}^{\nu}\left[e^{-r\tau}\bar{\varphi}(x\Gamma_{\nu}(\tau))\right]=\sup_{\tau\in\mathscr{S}}\mathbb{E}^{0}\left[e^{-r\tau}\bar{\varphi}(x\Gamma_{0}(\tau))\right].$$

Taking the supremum of the left-hand-side of (C.3) over $\nu(\cdot)\in\mathscr{D}$ and $\tau\in\mathscr{S}$, we deduce

$$V\leq\sup_{\tau\in\mathscr{S}}\mathbb{E}^{0}\left[e^{-r\tau}\bar{\varphi}(S(\tau))\right],$$

since $S(\cdot)=x\Gamma_0(\cdot)$. Thus, in order to prove (6.6) and the Theorem, it remains to show the reverse inequality

$$(C.4)\qquad V\geq\sup_{\tau\in\mathscr{S}}\mathbb{E}^{0}\left[e^{-r\tau}\bar{\varphi}(S(\tau))\right].$$

Now (C.4) is clearly satisfied if $V=\infty$, so let us concentrate on the case $V<\infty$. It suffices to prove, for every $0\leq t<T$, that

$$(C.5)\qquad \bar{X}(t)\geq\bar{\varphi}(S(t))\quad\text{holds a.s.}$$

for the process $\bar{X}(\cdot)$ of (5.23); because then the right-continuity of $\bar{X}(\cdot)$ and Lemmas A.1 and A.2 imply

$$V=\bar{X}(0)\geq\sup_{\tau\in\mathscr{S}}\mathbb{E}^{0}\left[e^{-r\tau}\bar{X}(\tau)\right]\geq\sup_{\tau\in\mathscr{S}}\mathbb{E}^{0}\left[e^{-r\tau}\bar{\varphi}(S(\tau))\right],$$

namely (C.4).

To prove (C.5), let us start by observing that, for every $0\leq t<T$, the inequalities

$$(C.6)\qquad \bar{X}(t)\geq\sup_{t<\theta\leq T}\bar{u}(t,\theta;S(t))\geq\underline{\lim}_{\theta\downarrow t}\bar{u}(t,\theta;S(t))\quad\text{hold a.s.},$$

where

$$\bar{u}(t,\theta;x)\stackrel{\Delta}{=}e^{-r(\theta-t)}\operatorname*{esssup}_{\nu(\cdot)\in\mathscr{D}_{t,\theta}}\mathbb{E}^{\nu}\left[\exp\left\{-\delta\left(\int_t^{\theta}\nu(s)\,ds\right)\right\}\varphi\left(xe^{-\int_t^{\theta}\nu(s)ds}\Gamma_{\nu}(t;\theta)\right)\right]$$
$$\Gamma_{\nu}(t;\theta)\stackrel{\Delta}{=}\exp\left[\sigma(W_{\nu}(\theta)-W_{\nu}(t))+\left(r-\sigma^2/2\right)(\theta-t)\right].$$

For any given $x\in(0,\infty)$, let $\{\nu_k\}\subseteq\tilde{K}$ be a maximizing sequence in (6.3), i.e.

$$(C.7)\qquad \bar{\varphi}(x)=\lim_k\uparrow\left[e^{-\delta(\nu_k)}\varphi\left(xe^{-\nu_k}\right)\right].$$

Fix $k\in N$, and define $\mu(\cdot)\in\mathscr{D}_{t,\theta}$, by setting $\mu(s)\equiv\frac{\nu_k}{\theta-t}$ for $t\leq s\leq\theta$, and $\mu(s)\equiv 0$ otherwise; then we have

$$\mathbb{E}^{\mu}\left[\exp\left\{-\delta\left(\int_t^{\theta}\mu(s)\,ds\right)\right\}\varphi\left(xe^{-\int_t^{\theta}\mu(s)ds}\Gamma_{\mu}(t;\theta)\right)\right]$$
$$= \mathbb{E}^{\mu}\left[e^{-\delta(\nu_k)}\varphi\left(xe^{-\nu_k}\Gamma_{\mu}(t;\theta)\right)\right] = \mathbb{E}^{0}\left[e^{-\delta(\nu_k)}\varphi\left(xe^{-\nu_k}\Gamma_{0}(t;\theta)\right)\right]$$

from (C.2), and

$$\underline{\lim}_{\theta\downarrow t}\mathbb{E}^{\mu}\left[\exp\left\{-\delta\left(\int_t^{\theta}\mu(s)\,ds\right)\right\}\varphi\left(xe^{-\int_t^{\theta}\mu(s)ds}\Gamma_{\mu}(t;\theta)\right)\right]$$
$$(C.8) \qquad = \underline{\lim}_{\theta\downarrow t}\mathbb{E}^{0}\left[e^{-\delta(\nu_k)}\varphi\left(xe^{-\nu_k}\Gamma_{0}(t;\theta)\right)\right] \geq e^{-\delta(\nu_k)}\varphi\left(xe^{-\nu_k}\right)$$

from Fatou's lemma. Taking the supremum of the expression on the right-hand-side of (C.8) with respect to k, we obtain

$$\underline{\lim}_{\theta\downarrow t}\bar{u}(t,\theta;x) \geq \bar{\varphi}(x)\,,\ \forall x \in (0,\infty)\,,$$

in conjunction with (C.7), and (C.5) follows now from this and from (C.6). □

Proof of Theorem 6.2: Equation (C.1) yields

$$\mathbb{E}^{\nu}\left[\exp\left\{-r\tau - \int_0^{\tau}\delta(\nu(s))\,ds\right\}\varphi(S(\tau))\right]$$
$$\geq \mathbb{E}^{\nu}\left[\exp\left\{-r\tau - \delta\left(\int_0^{\tau}\nu(s)\,ds\right)\right\}\varphi\left(xe^{-\int_0^{\tau}\nu(s)ds}\Gamma_{\nu}(\tau)\right)\right]$$
$$(C.9) \qquad \geq \mathbb{E}^{\nu}\left[e^{-r\tau}\underline{\varphi}(x\Gamma_{\nu}(\tau))\right]\ .$$

It follows from (C.9) and (C.2) that

$$\sup_{\tau\in\mathscr{S}}\mathbb{E}^{\nu}\left[\exp\left\{-r\tau - \int_0^{\tau}\delta(\nu(s))\,ds\right\}\varphi(S(\tau))\right]$$
$$\geq \sup_{\tau\in\mathscr{S}}\mathbb{E}^{\nu}\left[e^{-r\tau}\underline{\varphi}(x\Gamma_{\nu}(\tau))\right]$$
$$(C.10) \qquad = \sup_{\tau\in\mathscr{S}}\mathbb{E}^{0}\left[e^{-r\tau}\underline{\varphi}(x\Gamma_{0}(\tau))\right] = \sup_{\tau\in\mathscr{S}}\mathbb{E}^{0}\left[e^{-r\tau}\underline{\varphi}(S(\tau))\right]$$

holds for every $\nu(\cdot) \in \mathscr{D}$; taking the infimum on the left-hand-side of (C.10) over this class, we obtain (6.7). □

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