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On-Line Supplement to the Paper "Revenue Management of Callable Products" *

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In this on-line supplement to the paper "Revenue Management of Callable Products", we shall provide proofs of Lemma 1, Lemma 2, Lemma 3, Lemma 4, and Proposition 6 as well as Equation 11 rewritten for fast compuation.

Proof of Lemma 1.

On the set $\{D_L \leq a\}$, $S_L(a+1) = S_L(a) = D_L$, which means that $V_L(a+1)$ and $V_L(a)$ have the same distribution. Thus, $E[R(a+1) - R(a)|D_L \leq a] = 0$, from which we have

$$\Delta r(a,p) = E[R(a+1,p) - R(a,p)|D_L \geq a+1]P(D_L > a). \tag{1}$$

On the set $\{D_L \geq a+1\}$, $S_L(a+1) = a+1$ and $S_L(a) = a$. Since $E(X - Y)$ only depends on the marginal distribution of X and Y, and does not depend on the dependent structure of X and Y, we have

$$\begin{aligned} & E[\{\min(D_H, c - \bar{V}_L(a+1)) - \min(D_H, c - \bar{V}_L(a))\}|D_L \geq a+1] \\ = \quad & E[\{\min(D_H, c - \sum_{i=1}^{a+1} \bar{\xi}_i) - \min(D_H, c - \sum_{i=1}^{a} \bar{\xi}_i)\}|D_L \geq a+1] \\ = \quad & E[\min(D_H, c - \sum_{i=1}^{a+1} \bar{\xi}_i) - \min(D_H, c - \sum_{i=1}^{a} \bar{\xi}_i)], \end{aligned}$$

via the independence of D_L and D_H, where $\bar{\xi}_i$, $i \geq 1$, are independent Bernoulli random variables with a success probability $\bar{q}$. Since D_H is also an integer valued random variable and the values of

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$c - \sum_{i=1}^{a+1} \bar{\xi}_i$ and $c - \sum_{i=1}^{a} \bar{\xi}_i$ can only differ by at most 1, we have

$$\begin{aligned}
\min(D_H, c - \sum_{i=1}^{a+1} \bar{\xi}_i) - \min(D_H, c - \sum_{i=1}^{a} \bar{\xi}_i) &= \{c - \sum_{i=1}^{a+1} \bar{\xi}_i - (c - \sum_{i=1}^{a} \bar{\xi}_i)\} 1_{\{D_H \geq c - \sum_{i=1}^{a} \bar{\xi}_i\}} \\
&= -\bar{\xi}_{a+1} 1_{\{D_H \geq c - \sum_{i=1}^{a} \bar{\xi}_i\}}.
\end{aligned}$$

Therefore,

$$E[\{\min(D_H, c - \bar{V}_L(a+1)) - \min(D_H, c - \bar{V}_L(a))\} | D_L \geq a+1] = -\bar{q} P\left\{D_H \geq c - \sum_{i=1}^{a} \bar{\xi}_i\right\}. \quad (2)$$

Similarly,

$$\begin{aligned}
& E[\min((S_L(a+1) + D_H - c)^+, V_L(a+1)) - \min((S_L(a) + D_H - c)^+, V_L(a)) | D_L \geq a+1] \\
= \; & E[\{\min((a+1+D_H - c)^+, \sum_{i=1}^{a+1} \xi_i) - \min((a + D_H - c)^+, \sum_{i=1}^{a} \xi_i)\}] \\
= \; & P\left\{\xi_{a+1} = 0, \; (a+1+D_H-c)^+ > (a+D_H-c)^+, \; \sum_{i=1}^{a} \xi_i > (a+D_H-c)^+\right\} \\
& + P\left\{\xi_{a+1} = 1, \; (a+1+D_H-c)^+ = (a+D_H-c)^+, \; (a+1+D_H-c)^+ \geq \sum_{i=1}^{a+1} \xi_i\right\} \\
& + P\left\{\xi_{a+1} = 1, \; (a+1+D_H-c)^+ = (a+D_H-c)^+ + 1\right\} \\
= \; & \bar{q} P\left\{\sum_{i=1}^{a} \xi_i > a + D_H - c \geq 0\right\} + q P\{D_H \geq c - a\} \\
= \; & \bar{q}\left\{P\left\{\sum_{i=1}^{a} \xi_i > a + D_H - c \geq 0\right\} - P\{a + D_H - c \geq 0\}\right\} + P\{D_H \geq c - a\} \\
= \; & -\bar{q} P\left\{\sum_{i=1}^{a} \xi_i \leq a + D_H - c\right\} + P\{D_H \geq c - a\}.
\end{aligned}$$

In other words,

$$\begin{aligned}
& E[\min((S_L(a+1) + D_H - c)^+, V_L(a+1)) - \min((S_L(a) + D_H - c)^+, V_L(a)) | D_L \geq a+1] \\
= \; & -\bar{q} P\{c - \text{bino}(a, \bar{q}) \leq D_H\} + P\{D_H \geq c - a\}. \quad (3)
\end{aligned}$$

Combining (1), (2), and (3) yields

$$\begin{aligned}
& r(a+1, p) - r(a, p) \\
= \; & p_L - \bar{q} p_H P\{D_H \geq c - \text{bino}(a, \bar{q})\} + p\bar{q} P\{c - \text{bino}(a, \bar{q}) \leq D_H\} - pP\{D_H \geq c - a\},
\end{aligned}$$

which completes the proof.

Proof of Lemma 2.

Proof. We will first show that $a_T^* \leq a(p)$. It is enough to prove that $\psi(a,p) \leq p_H P\{D_H \geq c-a\}$. But this is equivalent to $(p_H - p)\bar{q}P\{D_H \geq c - \text{bino}(a,\bar{q})\} \leq (p_H - p)P\{D_H \geq c - a\}$, which holds because $\bar{q} = 1 - g(p) \leq 1$ and $P\{D_H \geq c - \text{bino}(a,\bar{q})\} \leq P\{D_H \geq c - a\}$. Intuitively, we do not need to protect more than b units of capacity for high fare customers and therefore at least $(c-b)^+$ units of capacity should be made available for sale at the low fare. To make this intuition more formal we observe that if $b < c$, then at $a = c - b$ we have $p_H P\{D_H \geq c - a\} = p_H P\{D_H \geq b\} = 0$. Thus, by the definition of a_T^*, we have $a_T^* \geq c - b$, completing the proof. $\square$

Proof of Lemma 3.

We first note the following properties of the incomplete beta function:

$$I_q(a, n-a+1) = \sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j}, \quad 0 \leq a \leq n+1; \quad I_x(a,b) = 1 - I_{1-x}(b,a);$$

$$\frac{d}{dq} I_q(a,b) = \frac{q^{a-1}(1-q)^{b-1}}{B(a,b)}, \quad B(a+1,b) = \frac{a}{a+b} B(a,b), \quad a,\ b > 0. \tag{4}$$

We shall only study the case when $n \geq 2$ and $1 \leq y \leq n-1$, as the other two cases hold automatically. In this case, when $y \geq 1$,

$$\begin{aligned} E[\min(X,y)] &= \sum_{i=1}^{y} i \binom{n}{i} q^i (1-q)^{n-i} + yP\{X \geq y+1) \\ &= nq \sum_{i=1}^{y} \binom{n-1}{i-1} q^{i-1} (1-q)^{n-i} + yP\{X \geq y+1) \\ &= nq \sum_{j=0}^{y-1} \binom{n-1}{j} q^j (1-q)^{n-1-j} + yP\{X \geq y+1) \\ &= nq\{1 - I_q(y, n-y)\} + yI_q(y+1, n-y). \end{aligned}$$

We also have

$$\begin{aligned} \frac{d}{dq} E[\min(X,y)] &= n\{1 - I_q(y,n-y)\} + nq\{-\frac{q^{y-1}(1-q)^{n-y-1}}{B(y,n-y)}\} + y\frac{q^y(1-q)^{n-y-1}}{B(y+1,n-y)} \\ &= n\{1 - I_q(y,n-y)\} + nq\{-\frac{q^{y-1}(1-q)^{n-y-1}}{B(y,n-y)}\} + n\frac{q^y(1-q)^{n-y-1}}{B(y,n-y)} \\ &= n\{1 - I_q(y,n-y)\}, \end{aligned}$$

via (4). Finally, when $y \geq x$, $x \in [0, n]$, and $n \geq 1$, we have

$$P\{\min(X, y) > x) = P\{X > x\} = \sum_{j=\lfloor x \rfloor + 1}^{n} \binom{n}{j} q^j (1-q)^{n-j} = I_q(\lfloor x \rfloor + 1, n - \lfloor x \rfloor),$$

which completes the proof. □

Proof of Lemma 4.

Proof. By Assumption 2, it is sufficient to show that $\frac{d}{dq}E[\min(G(a), V_L(a))]$ is positive and strictly decreasing in q, $q \in [0, \bar{q}_H]$, because then $E[\min(G(a), V_L(a))]$ must be strictly increasing in q. To do this, note that by Lemma 3

$$\begin{aligned}\frac{d}{dq}E[\min(G(a), V_L(a))] &= E[S_L(a)\{1 - I_q(G(a), S_L(a) - G(a))\};\ S_L(a) + D_H > c,\ D_H \leq c - 1] \\ &\quad + E[S_L(a);\ D_H \geq c,\ S_L(a) \geq 1].\end{aligned}$$

On the set $\{S_L(a) + D_H > c,\ D_H \leq c - 1\}$, by the definition of $G(a)$, we must have $G(a) > 0$, $S_L(a) < G(a)$. Thus, $0 < I_q(G(a), S_L(a) - G(a)) < 1$, and $I_q(G(a), S_L(a) - G(a))$ is strictly increasing in q, for $0 < q < 1$, by (10). Since $E[X] > 0$ for any random variable $X \geq 0$ with $P\{X > 0\} > 0$, Assumption 1 implies that $\frac{d}{dq}E[\min(G(a), V_L(a))]$ must be positive and strictly decreasing in q, $q \in [0, \bar{q}_H]$. To check for uniqueness, notice that the left side of (9) is a strictly increasing function of q, the right side is a strictly decreasing function of q, and when $q = 0$ (at the recall price p_L) the left hand side is zero (because $V_L(a) = 0$). If at $q = \bar{q}_H$ the right side of (9) is not greater than the left side, then there must be a unique root within $[0, \bar{q}_H]$; otherwise, $\frac{d}{dq}r(a, p) > 0$ for all $p \in [p_L, \bar{q}_H]$, and $\bar{q}_H$ must be optimal. □

Proof of Proposition 6.

Consider the case of a risk-neutral first-period customer with maximum willingness-to-pay of R_L who is faced with the choice between purchasing a low-fare standard product for p_L or a pure callable with a price of p_C and a call price of $\hat{p}$. He will purchase the standard product if $R_L - p_L \geq \max[(1-q)(R_L - p_c) + q(p - p_c), 0]$ and the callable product if $(1-q)(R_L - p_c) + q(p - p_c) > \max[R_L - p_L, 0]$, where $q > 0$ is his *ex ante* probability that the airline will exercise the call for his product. Assume that, instead of offering callables at a price of p_C and a strike price of p along with the standard low-price products, the airline offers both callables and standard products at the price p_L with a free callable option at $\hat{p}$. Then, a customer with valuation R_L would purchase the call option in both cases if

$$(1-q)(R_L - p_C) + q(p - p_C) = (1-q)(R_L - p_L) + q(\hat{p} - p_L)$$

or $\hat{p} = p + (p_L - p_c)/q$. Note that $\hat{p}$ is independent of R_L. This means that offering both standard and callable products at p_L with a strike price of $\hat{p}$ will result in the same demand for any distribution of R_L as offering standard products at p_L and callable products at p_C with a strike price of p assuming only that customers are risk neutral and share a common *ex ante* probability q.

We now need to show that the provider will achieve the same expected revenue in both cases. Let $R(\alpha)$ be revenue in the case when callables cost p_C and the exercise price is p and $\hat{R}(\alpha)$ be the case when callables cost p_L and the exercise price is $\hat{p} = p + (p_L - p_C)/q$. Then, using the notation of Equation (1):

$$\begin{aligned}
R(\alpha) &= p_C V_L(\alpha) + p_L[S_L(\alpha) - V_L(\alpha)] + p_H \min(D_H, c - \bar{V}_L(\alpha)] - p \min[(S_L(\alpha) + D_H - c)^+, V_L(\alpha)] \\
\hat{R}(\alpha) &= p_L S_L(\alpha) + p_H \min[D_H, c - \bar{V}_L(\alpha)] - p \min[(S_L(\alpha) + D_H - c)^+, V_L(\alpha)],
\end{aligned}$$

so that

$$\begin{aligned}
R(\alpha) - \hat{R}(\alpha) &= p_C V_L(\alpha) + p_L(S_L(\alpha) - V_L(\alpha)) - p_L S_L(\alpha) + (\hat{p} - p) \min[S_L(\alpha) + D_H - c), V_L(\alpha)] \\
&= (p_C - p_L) V_L(\alpha) + (\hat{p} - p) \min[S_L(\alpha) + D_H - c, V_L(\alpha)], \\
&= (p_C - p_L) V_L(\alpha) + \frac{p_L - p_C}{q} \min[S_L(\alpha) + D_H - c, V_L(\alpha)].
\end{aligned}$$

If each customer correctly anticipates the fraction of products that will be called, then

$$q = \frac{\min[S_L(\alpha) + D_H - c, V_L(\alpha)]}{V_L(\alpha)}$$

and $R(\alpha) - \hat{R}(\alpha) = 0$.

Rewriting the Terms in Equation 11 for Programming Purposes.

$$\begin{aligned}
& E[S_L(a) I_{\bar{q}}(S_L(a) - G(a), G(a)); S_L(a) + D_H > c, D_H \le c - 1] \\
= & \sum_{i=2}^{a} \sum_{j=0}^{c-1} P\{D_L = i, D_H = j\} i I_{\bar{q}}(c - j, i + j - c) \mathbf{1}_{\{i+j \ge c+1\}} \\
& \quad + \sum_{j=0}^{c-1} P\{D_L \ge a + 1, D_H = j\} a I_{\bar{q}}(c - j, a + j - c) \mathbf{1}_{\{a+j \ge c+1\}},
\end{aligned}$$

$$
\begin{aligned}
& E[G(a)I_q(G(a)+1, S_L(a)-G(a)); S_L(a)+D_H > c, D_H \le c-1] \\
= & \sum_{i=2}^{a}\sum_{j=0}^{c-1} P\{D_L = i, D_H = j\}(i+j-c)I_q((i+j-c+1, c-j)\mathbf{1}_{\{i+j\ge c+1\}} \\
& + \sum_{j=0}^{c-1} P\{D_L \ge a+1, D_H = j\}(a+j-c)I_q(a+j-c+1, c-j)\mathbf{1}_{\{a+j\ge c+1\}},
\end{aligned}
$$

$$
E[S_L(a); D_H \ge c, S_L(a) \ge 1] = \sum_{i=0}^{a} P\{D_L = i, D_H \ge c\}i + a\sum_{i=a+1}^{\infty} P\{D_L = i, D_H \ge c\}.
$$

Also, for traditional revenue management, we have

$$
\begin{aligned}
r(a) &= p_L E[S_L(a)] + p_H E[\min(c - S_L(a), D_H)] \\
&= p_L \left\{ \sum_{i=0}^{a} P\{D_L = i)i + aP\{D_L \ge a+1\} \right\} \\
&\quad + p_H \left\{ \sum_{i=0}^{a}\sum_{j=0}^{c-i} P\{D_L = i, D_H = j\}j + \sum_{i=0}^{a}(c-i)P\{D_L = i, D_H \ge c-i+1\} \right. \\
&\quad \left. + P\{D_L \ge a+1\} \left[\sum_{j=0}^{c-a} P\{D_H = j\}j + (c-a)P\{D_H \ge c-a+1\} \right] \right\}.
\end{aligned}
$$